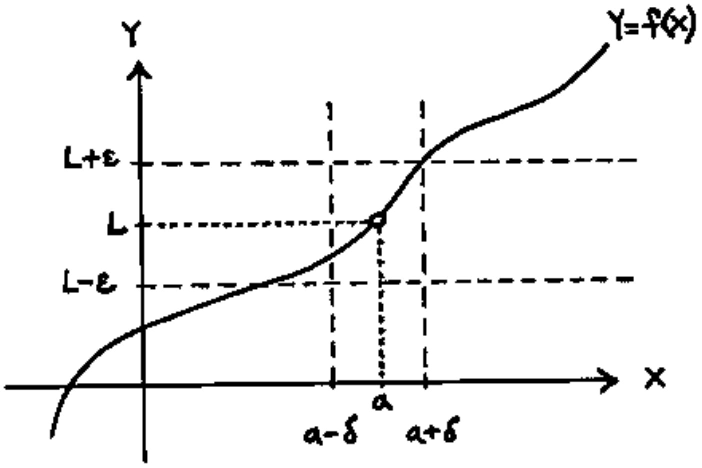
Teaching Notes for Calculus
Homework #1

Definition of a Limit: The statement $\lim_{x \to a} f(x) = L$ has the following precise definition. Given any real number $\varepsilon > 0$, there exists another real number $\delta > 0$ so that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

The best way to explain this definition is by drawing the graph below. What this picture



shows is a hole in the function which means that the function is undefined when $x = a$. And, even though it appears that y might be L when $x = a$, it can't be since the function is undefined when $x = a$. This is what the whole concept of a limit is all about! It asks the following question: If you can't actually evaluate the function at $x = a$, where is the function going, or approaching, on the y axis, as x gets closer and closer to a? And the answer to this question can be complicated, as you can approach a from both the left and the right. On the graph, this idea is demonstrated by the $a - \delta$ and the $a + \delta$. As δ gets smaller and smaller, you get closer and closer to a on the x axis. Since the entire point of this exercise is to find out what L is, the limit on the y axis, as you get closer and closer to a, you need to set up bounds on the y axis, around L. On the graph, this idea is demonstrated by the $L - \varepsilon$ and the $L + \varepsilon$. As ε gets smaller and smaller, you get closer and closer to L on the x axis. The trick to proving that this will actually happen on any given function is finding a relationship between ε and δ that guarantees that as $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Proof of a Limit Steps:

Step 1: Find the exact answer to the limit (L) through graphing and substitution.
Step 2: State that $0 < |x - a| < \delta$ where $a =$ where x is going to (substitute in a) gives you $|x - a| < \delta$ Step 3: State that $|f(x) - L| < \varepsilon$ and substitute in f(x) and L
Step 4: Use any legitimate math manipulations to turn $|f(x) - L|$ into $|x - a|$
Step 5: Once they are equal, you will have a relationship between ε and δ

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Step 6: Now complete the proof: If $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$ by basically working backward starting with $0<|x-a|<\delta$ and ending up with $|f(x)-L|<\varepsilon$

Example: Find and Prove $\lim_{x\to 2}(3x+5)=L$

Step 1: $L=11$
Step 2: Find a relationship between ε and δ by stating that $0<|x-a|<\delta$ where $a=2$ which gives you $|x-2|<\delta$ by substitution.
Step 3: State that $|f(x)-L|<\varepsilon$. Substitution yields $|(3x+5)-(11)|<\varepsilon$.
Step 4: Simplify to get $|3x-6|<\varepsilon$ and then manipulate this inequality to get $|3(x-2)|<\varepsilon$ which is equivalent to $3|(x-2)|<\varepsilon$ and finally $|(x-2)|<\frac{\varepsilon}{3}$.
Step 5: This means that the relationship between δ and ε is $\delta=\frac{\varepsilon}{3}$.
Step 6: Therefore, if $0<|x-2|<\delta$, then $0<|x-2|<\frac{\varepsilon}{3}$ then $3|x-2|<\varepsilon$ then $|3(x-2)|<\varepsilon$ which can then be written as $|3x-6|<\varepsilon$ or $|3x+5-11|<\varepsilon$ and finally as $|(3x+5)-(11)|<\varepsilon$

You can also estimate a limit and, many times find it exactly, by graphing the function, creating a table of decimal values starting at .1 below what x is approaching, then .01 below, then .001 below. Finish the table by going .001 above what x is approaching, then .01 above, and finally .1 above. The answers you get for the table should converge as you head to .001 from the right and as you head to .001 from the left. Many times, those two answers will be the same while other times they will be very close to each other. Your estimate to the limit would be either of the answers from the .001 columns. If the answer is exact, you should write it as a fraction. If the answers diverge (the .001 columns are far apart) the answer is that the limit does not exist!

Direct substitution can also help to determine the exact answers as long as the limit isn't undefined at that value or if the answer to the substitution gives you a different value than the two converging .001 columns.

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Classroom Examples

1) Graph, and then use the graph, along with a detailed table of values, to find

$\lim_{x\to -1}\frac{x+1}{x^2-2x-3}$, if it exists.

Answer: The graph appears to be approaching $-.025$ or $\frac{-1}{4}$ (Don't forget to talk about the hole in the graph!) and the table of values appears to be converging to $-.025$. In this problem, you need to be a bit creative in order to confirm your answer through substitution as $\frac{x+1}{x^2-2x-3}$ is undefined at $x=-1$. In this case, factoring and doing a War Game would be helpful. This results in $\lim_{x\to -1}\frac{1}{x-3}$. Substitution now results in the final answer of

$\lim_{x\to -1}\frac{1}{x-3}=\frac{-1}{4}$.

2) Graph, and then use the graph, along with a detailed table of values, to find

$\lim_{x\to 3}\frac{x-4}{x^2-7x+12}$, if it exists.

Answer: The graph appears to have a serious issue at $x=3$. The table of values confirms this as the answers to the .001 columns are dramatically diverging. Substitution, even if you factor and kill stuff, gives you an undefined answer. Therefore

$\lim_{x\to 3}\frac{x-4}{x^2-7x+12}=$ Does Not Exist.

3) Graph, and then use the graph, along with a detailed table of values, to find

$\lim_{x\to 4}\frac{x-4}{x^2-7x+12}$, if it exists.

Answer: Again, this graph appears to have a serious issue at $x=3$ (Again, make sure that the students find and draw the hole at $x=4$) but we only care about the graph as x

approaches 4. Factoring and killing stuff results in $\lim_{x\to 4}\frac{1}{x-3}$. Substitution gives us a solid answer of 1. The table of values confirms this as the answer as the .001 columns are converging to 1. Therefore, $\lim_{x\to 4}\frac{x-4}{x^2-7x+12}=1$.

4) Graph, and then use the graph, along with a detailed table of values, to find:

$$\lim_{x\to 3} f(x) \text{ if } f(x)=\begin{cases} 4 & x\neq 3 \\ -5+x & x=3 \end{cases}$$, if it exists.

Answer: This piece-wise graph appears to be approaching 4 (Make sure that students not only draw the hole in the line, but that they also remember to place a solid dot at -2 and the table of values appears to be converging to 4. Explain that even though $f(3)=-2$, the limit as you approach 3 from both sides equals 4. In this problem, substitution would result in a VERY wrong answer which demonstrates that you can't just rely on one method…always examine the graph, create a table, and try substitution. Then you can make a final decision that makes sense! Based on the graph and the converging table of values, $\lim_{x\to 3} f(x)=4$

5) Graph, and then use the graph, along with a detailed table of values, to find $\lim_{x\to -3}\frac{x-3}{x^2-9}$, if it exists.

Answer: The graph appears to have a serious issue at $x=3$ (Again, make sure that the students find and draw the hole at $x=-3$), but we only care about the graph as x approaches -3. The graph appears to show the function approaching -3.25 as x approaches -3. Factoring and killing stuff results in $\lim_{x\to -3}\frac{1}{x+3}$. Substitution gives us an answer of undefined, so that doesn't really help. The table of values confirms that the .001 columns are converging to the answer -3.25. Therefore, $\lim_{x\to -3}\frac{x-3}{x^2-9}=-3.25=\frac{-13}{4}$.

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6) Graph, and then use the graph, along with a detailed table of values, to find $\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{-x}$, if it exists.

Answer: The graph appears to have a serious issues when $x<-2$ because the function goes to imaginary land when $x<-2$ (Again, make sure that the students find and draw the hole at $x=0$), but we only care about the graph as x approaches 0. The graph appears to show the function approaching approximately $-.35$ as x approaches 0. Substitution gives us an answer of $\frac{0}{0}$ which is undefined, so that doesn't really help. The table of values confirms that the .001 columns are converging to somewhere between $.3535$ and $.3536$. Therefore, $\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{-x}=-.3540$ or $-.3536$.

7) Find L, the $\lim_{x\to 5}(-2x+3)$, and then find $\delta>0$ such that $|f(x)-L|<.001$ whenever $0<|x-a|<\delta$.

These are basically proofs that stop at step 5 (finding the relationship between ε and δ and then plug in the value for ε

Answer: $L=-7$ Find a relationship between ε and δ by stating that $0<|x-a|<\delta$ where $a=5$ which gives you $|x-(5)|<\delta$ by substitution, which simplifies to $|x-5|<\delta$. Now state that $|f(x)-L|<\varepsilon$. Substitution yields $|(-2x+3)-(-7)|<\varepsilon$ which simplifies to $|-2x+10|<\varepsilon$. Factor to get $|-2(x-5)|<\varepsilon$ and then manipulate to get $2|x-5|<\varepsilon$ which results in $|x-5|<\frac{\varepsilon}{2}$ which means that the relationship between δ and ε is $\delta=\frac{\varepsilon}{2}$. If $\varepsilon=.001$, then $\delta=\frac{.001}{2}=.0005$.

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8) Find L, the $\lim_{x \to -1}(x^2+4)$, and then find $\delta > 0$ such that $|f(x)-L| < .01$ whenever $0 < |x-a| < \delta$.

Answer: $L=5$ Find a relationship between ε and δ by stating that $0<|x-a|<\delta$ where $a=-1$ which gives you $|x-(-1)|<\delta$ by substitution, which simplifies to $|x+1|<\delta$. Now state that $|f(x)-L|<\varepsilon$. Substitution yields $|(x^2+4)-(5)|<\varepsilon$ which simplifies to $|x^2-1|<\varepsilon$. Factor to get $|(x+1)(x-1)|<\varepsilon$. In a situation like this, it's impossible to manipulate the left side of the inequality to get some multiple of $|x+1|<\varepsilon$. In a situation like this, it is permissible to set up bounds for your proof to eliminate the $(x-1)$. If you are trying to evaluate the limit as x approaches -1, choose the smallest possible integer bounds around -1. In this case, choose $[-2,0]$ and then chose the boundary that makes $|x-1|$ the greatest. In this case, substituting -2 into $|x-1|$ gives you the greatest answer of 3. Therefore, on the interval $[-2,0]$, $|(x+1)(x-1)|<\varepsilon$ is equivalent to $3|(x+1)|<\varepsilon$ which results in $|x+1|<\frac{\varepsilon}{3}$ which means that the relationship between δ and ε is $\delta=\frac{\varepsilon}{3}$. If $\varepsilon=.01$, then $\delta=\frac{.01}{3}=\frac{1}{300}=.00\bar{3}$.

9) Find L, the $\lim_{x \to -4}(2x+10)$, and then use the definition of a limit to prove that the limit is L.

Answer: $L=2$ Proof: Find a relationship between ε and δ by stating that $0<|x-a|<\delta$ where $a=-4$ which gives you $|x-(-4)|<\delta$ by substitution which simplifies to $|x+4|<\delta$. Now state that $|f(x)-L|<\varepsilon$. Substitution yields $|(2x+10)-(2)|<\varepsilon$. Simplify to get $|2x+8|<\varepsilon$ you must now manipulate this inequality to get $|2(x+4)|<\varepsilon$ which is equivalent to $2|(x+4)|<\varepsilon$ and finally $|x-4|<\frac{\varepsilon}{2}$. This means that the relationship between δ and ε is $\delta=\frac{\varepsilon}{2}$. Therefore, if $0<|x+4|<\delta$, then $0<|x+4|<\frac{\varepsilon}{2}$ then $2|x+4|<\varepsilon$ then $|2(x+4)|<\varepsilon$ which can then be written as $|2x+8|<\varepsilon$ or $|(2x+10)-2|<\varepsilon$ and finally as $|(2x+10)-(2)|<\varepsilon$

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10) Find L, the $\lim_{x\to 0}\sqrt[3]{3x}$, and then use the definition of a limit to prove that the limit is L.

Answer: $L = 0$ Proof: Find a relationship between ε and δ by stating that $0 < |x - a| < \delta$ where $a = 0$ which gives you $|x - 0| < \delta$ by substitution. Now state that $|f(x) - L| < \varepsilon$. Substitution yields $\left|\sqrt[3]{3x} - 0\right| < \varepsilon$. Simplify to get $\left|\sqrt[3]{3x}\right| < \varepsilon$ you must now manipulate this inequality to get $\sqrt[3]{|3x|} < \varepsilon$ which can be written as $|3x| < \varepsilon^3$ which is equivalent to $3|x - 0| < \varepsilon^3$ and finally as $|x - 0| < \frac{\varepsilon^3}{3}$. This means that the relationship between δ and ε is $\delta = \frac{\varepsilon^3}{3}$. Therefore, if $0 < |x - 0| < \delta$, then $0 < |x - 0| < \frac{\varepsilon^3}{3}$ which can become $3|x - 0| < \varepsilon^3$ and can be further simplified to $\sqrt[3]{|3x - 0|} < \varepsilon$ which simplifies to $\left|\sqrt[3]{3x - 0}\right| < \varepsilon$ which can then be written as $\left|(\sqrt[3]{3x}) - (0)\right| < \varepsilon$.