Teaching Notes for Calculus
Homework #10
Newton's Method of Approximation and L`Hopital's Rule for Limits

Begin the lesson by adding two new derivative rules that deal with exponentials and logarithmics: If $f(x)=e^x \rightarrow f'(x)=e^x$ and if $f(x)=\ln x \rightarrow f'(x)=\frac{1}{x}$. Mention the amazing fact that e^x is the only function whose derivative is itself! This means that the answer to e to any exponent is the slope on the graph of e^x at that same point!

Newton's Method of Approximation

When you graph a polynomial equation, especially REALLY complicated ones, the graph might hit the x axis in many places (lots of zeros). However, unless the problem is contrived ahead of time to work out just right, it is impossible to find all of these roots. Thanks to Newton, however, as long as we have made the graph and have an integer very close to the unknown root, you can find the approximate answer to the root that will be perfectly accurate to as many decimal places as you choose to work out.

Newton's Method for approximating solutions to polynomial equations:

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

Prove Newton's Method by starting with two points (x_{n+1},y_{n+1}) and (x_n,y_n), write the slope formula using these two points, and then solve for x_{n+1} when $y_{n+1}=0$ and $m=f'(x_n)$

Proof: Choose two points, (x_{n+1},y_{n+1}) and (x_n,y_n) and use $m=\frac{y_1-y_2}{x_1-x_2}$ and substitution to get $f'(x_n)=\frac{y_n-y_{n+1}}{x_n-x_{n+1}}$. Solving for x_{n+1} yields $x_{n+1}=x_n-\frac{y_n-y_{n+1}}{f'(x_n)}$. Since we are looking for a more exact answer for a zero of the function, make $y_{n+1}=0$ and rewrite y_n as $f(x_n)$ gives you $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$.

Important Note – Newton's Method can be used to find exact roots. You know you have found an exact root when the answers to two successive iterations give you the same answer.

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Four Steps for Implementing Newton's Method

Step 1) Find the derivative

Step 2) Based on your knowledge of the function and/or a sketch of the graph, make an initial guess as to the solution and call it x_1 (in general, choose the closest integer to the zero you are trying to find. *IMPORTANT limitation of Newton's Method* Nothing "bad" can happen between the guess and the actual answer. For example, if the derivative of the function becomes zero or is undefined at any point between your initial guess and the actual answer, Newton's Method formula becomes undefined and will therefore fail!

Step 3) Use $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ to calculate a better answer than the initial guess

Step 4) Continue this process for as many iterations as desired

L'Hopital's Rule

This rule gives you an extra tool for trying to determine the limit for two types of indeterminate forms, $\pm\dfrac{0}{0}$ and $\pm\dfrac{\infty}{\infty}$. There are other indeterminate forms for limits where you can't apply the rule: $\infty - \infty, 0 \cdot \infty, 0^0, 1^\infty, \infty^0$ *Note – the reason that you didn't learn about this method while we where studying limits is that this method requires differentiation*

L'Hopital's Rule helps to find limits for the indeterminate forms $\frac{0}{0}$ and $\pm\frac{\infty}{\infty}$, provided that the functions $f(x)$ and $g(x)$ are both differentiable on some open interval surrounding the limit point, c, except possibly at c itself. *Be VERY careful while attempting these types of problems as L'Hopital's Rule only works on quotients. However, there are many mathematical manipulations which can turn the other indeterminate forms into quotients*

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L'Hopital's Rule: $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$ *You can apply this rule multiple times!

Classroom Exercises:

1) Use Newton's Method to find the exact root for $f(x) = mx + b$

Answer:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad x_2 = x_1 - \frac{mx_1 + b}{m} = x_1 - x_1 - \frac{b}{m} = -\frac{b}{m} = x_2$$

$$x_3 = x_2 - \frac{mx_2 + b}{m} = x_2 - x_2 - \frac{b}{m} = -\frac{b}{m} = x_3$$

2) Use Newton's Method to find the exact root for $f(x) = \frac{-3}{7}x + 2$ using $x_1 = 5$ as your first guess. Answer:

$$x_2 = 5 - \frac{\frac{-3}{7}(5) + 2}{\frac{-3}{7}} = 5 - 5 + \frac{14}{3} = \frac{14}{3} = x_2 \qquad x_3 = \frac{14}{3} - \frac{\frac{-3}{7}(\frac{14}{3}) + 2}{\frac{14}{3}} = \frac{14}{3} - 0 = \frac{14}{3} = x_3$$

2) Use Newton's Method to find the exact root for $f(x) = ax^2 + bx + c$

Answer: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$x_2 = x_1 - \frac{ax_1^2 + bx_1 + c}{2ax_1 + b} = x_1 - \frac{x_1}{2} - \frac{bx_1 + 2c}{2(2ax_1 + b)} = \frac{x_1}{2} - \frac{bx_1 + 2c}{2(2ax_1 + b)} = x_2$$

$$x_3 = x_2 - \frac{ax_2^2 + bx_2 + c}{2ax_2 + b} = \left(\frac{x_1}{2} - \frac{bx_1 + 2c}{2(2ax_1 + b)}\right) - \frac{ax_2^2 + bx_2 + c}{2ax_2 + b} = \left(\frac{x_1}{2} - \frac{bx_1 + 2c}{2(2ax_1 + b)}\right) - \frac{x_2}{2} - \frac{bx_2 + 2c}{2(2ax_2 + b)} = x_3$$

3) Use the function $f(x) = -2x^2 + 5x + 6$ and Newton's Method, with an initial guess of $x_1 = 3$. Do as many iterations as it takes for your approximation to be correct to six decimal places.

Answer:

$$x_2 = 3 - \frac{-2(3)^2 + 5(3) + 6}{-2(2)(3) + 5} = 3 + \frac{3}{7} = \frac{24}{7} = x_2 \approx 3.428571$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{24}{7} - \frac{-2\left(\frac{24}{7}\right)^2 + 5\left(\frac{24}{7}\right) + 6}{2(-2)\left(\frac{24}{7}\right) + 5} = \frac{24}{7} - \frac{\frac{-18}{49}}{\frac{-61}{7}} = \frac{24}{7} - \frac{18}{427} = \frac{1446}{427} = x_3 \approx 3.386417$$

$$x_4 = x_3 - \frac{f(x_2)}{f'(x_2)} = \frac{1446}{427} - \frac{-2\left(\frac{1446}{427}\right)^2 + 5\left(\frac{1446}{427}\right) + 6}{2(-2)\left(\frac{1446}{427}\right) + 5} = \frac{1446}{427} - \frac{648}{1558123} = \frac{5275806}{1558123} = x_4 \approx 3.386001$$

5) For the initial guess, $x_1 = 2$, complete two iterations of Newton's Method for the function $f(x) = -2x^3 + 3x^2 + 5$

Answer: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \to x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \to x_2 = 2 - \frac{-2(2)^3 + 3(2)^2 + 5}{-6(2)^2 + 6(2)} = 2 + \frac{1}{12} = \frac{25}{12} \approx 2.0833$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \to x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \to x_3 = \frac{25}{12} - \frac{-2\left(\frac{25}{12}\right)^3 + 3\left(\frac{25}{12}\right)^2 + 5}{-6\left(\frac{25}{12}\right)^2 + 6\left(\frac{25}{12}\right)} = \frac{25}{12} - \frac{11}{2340} = \frac{1216}{585} \approx 2.0786$$

6) Sketch a rough graph to find initial guesses and then approximate the zeros of the function $f(x) = 2x^4 - 5x^3 + 6x - 7$ using Newton's Method and continue the process until two successive approximations differ by less than .002.

Answers: From the graph, it appears that there are 2 real zeros, one close to –1 and the other close to +2. Therefore, you will have to use Newton's Method twice:

For $x_1 = -1$

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$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \to x_2 = -1 - \frac{2(-1)^4 - 5(-1)^3 + 6(-1) - 7}{8(-1)^3 - 15(-1)^2 + 6} = -1 - \frac{6}{17} = -\frac{23}{17} \approx -1.3529$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \to x_2 = \frac{-23}{17} - \frac{2\left(\frac{-23}{17}\right)^4 - 5\left(\frac{-23}{17}\right)^3 + 6\left(\frac{-23}{17}\right) - 7}{8\left(\frac{-23}{17}\right)^3 - 15\left(\frac{-23}{17}\right)^2 + 6} = \frac{-23}{17} + \frac{331236}{3446801} = -\frac{4332083}{3446801} \approx -1.2568$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \to x_3 = -\frac{4332083}{3446801} - \frac{2\left(-\frac{4332083}{3446801}\right)^4 - 5\left(-\frac{4332083}{3446801}\right)^3 + 6\left(-\frac{4332083}{3446801}\right) - 7}{8\left(-\frac{4332083}{3446801}\right)^3 - 15\left(-\frac{4332083}{3446801}\right)^2 + 6} \approx -1.2568 + .0112 \approx -1.2456$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} \to x_5 = -1.2456 - \frac{2(-1.2456)^4 - 5(-1.2456)^3 + 6(-1.2456) - 7}{8(-1.2456)^3 - 15(-1.2456)^2 + 6} \approx -1.2456 + .0001 \approx -1.2455$$

For $x_1 = 2$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \to x_2 = 2 - \frac{2(2)^4 - 5(2)^3 + 6(2) - 7}{8(2)^3 - 15(2)^2 + 6} = 2 + \frac{3}{10} = \frac{23}{10} \approx 2.3$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \to x_2 = \frac{23}{10} - \frac{2\left(\frac{23}{10}\right)^4 - 5\left(\frac{23}{10}\right)^3 + 6\left(\frac{23}{10}\right) - 7}{8\left(\frac{23}{10}\right)^3 - 15\left(\frac{23}{10}\right)^2 + 6} = \frac{23}{10} + \frac{27}{335} = \frac{1487}{670} \approx 2.2194$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \to x_3 = \frac{1487}{670} - \frac{2\left(\frac{1487}{670}\right)^4 - 5\left(\frac{1487}{670}\right)^3 + 6\left(\frac{1487}{670}\right) - 7}{8\left(\frac{1487}{670}\right)^3 - 15\left(\frac{1487}{670}\right)^2 + 6} = \frac{1487}{670} - \frac{9134473518}{985971325645} = \frac{4358266459633}{1971942651290} \approx 2.2101$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} \to x_4 = \frac{4358266459633}{1971942651290} - \frac{2\left(\frac{4358266459633}{1971942651290}\right)^4 - 5\left(\frac{4358266459633}{1971942651290}\right)^3 + 6\left(\frac{4358266459633}{1971942651290}\right) - 7}{8\left(\frac{4358266459633}{1971942651290}\right)^3 - 15\left(\frac{4358266459633}{1971942651290}\right)^2 + 6} =$$

$$\frac{4358266459633}{1971942651290} - \frac{8338499717169391617264063640230083193564095448}{7218876550644976712289886706638012942969360467008 5} =\approx 2.2100$$

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7) Use the function $f\left(x\right)=2x^3+5x^2-4x-10$ and Newton's Method to approximate $\sqrt{2}$ to five decimal places.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \to x_2 = 1 - \frac{2(1)^3+5(1)^2-4(1)-10}{6(1)^2+10(1)-4} = 1 + \frac{7}{12} = \frac{19}{12} \approx 1.58333$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \to x_3 = \frac{19}{12} - \frac{2\left(\frac{19}{12}\right)^3 + 5\left(\frac{19}{12}\right)^2 - 4\left(\frac{19}{12}\right) - 10}{6\left(\frac{19}{12}\right)^2 + 10\left(\frac{19}{12}\right) - 4} = \frac{19}{12} - \frac{3577}{23220} = \frac{8297}{5805} \approx 1.42929$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \to x_4 = \frac{8297}{5805} - \frac{2\left(\frac{8297}{5805}\right)^3 + 5\left(\frac{8297}{5805}\right)^2 - 4\left(\frac{8297}{5805}\right) - 10}{6\left(\frac{8297}{5805}\right)^2 + 10\left(\frac{8297}{5805}\right) - 4} = \frac{8297}{5805} - \frac{63772894559528}{4268606343509835} \approx 1.414345$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} \to x_5 = 1.414345 - \frac{2(1.414345)^3 + 5(1.414345)^2 - 4(1.414345) - 10}{6(1.414345)^2 + 10(1.414345) - 4} \approx 1.41421$$

8) Evaluate $\lim\limits_{x\to 2}\frac{3x^2-13x+14}{x^2-4}$, using techniques you learned for your first test and then evaluate it again using L'Hopital's Rule.

Answers: Using manipulations, from the first test, you get

$$\lim_{x\to 2}\frac{3x^2-13x+14}{x^2-4} \to \lim_{x\to 2}\frac{(3x-7)(x-2)}{(x-2)(x+2)} \to$$

$\lim\limits_{x\to 2}\frac{3x-7}{x+2} = \frac{-1}{4}$. Using L'Hopital's Rule, $\lim\limits_{x\to 2}\frac{f(x)}{g(x)} = \lim\limits_{x\to c}\frac{f'(x)}{g'(x)} \to \lim\limits_{x\to 2}\frac{6x-13}{2x} \to \frac{-1}{4}$

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9) Evaluate $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{x}$ using L'Hopital's Rule if necessary.

Answer: While L'Hopital's Rule can be used under any circumstances, you should always default to this method when you get $\frac{0}{0}$ through direct substitution. Using L'Hopital's Rule once gives you: $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{x}=\lim\limits_{x\to 0}\dfrac{2e^{2x}}{1}=2$

10) Evaluate $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cos(5x)}{\cos(-7x)}$ using L'Hopital's Rule if necessary.

Answer: Once again, you should always default to L'Hopital's Rule when you get $\frac{0}{0}$ through direct substitution. Using L'Hopital's Rule once gives you:

$$\lim_{x\to\frac{\pi}{2}}\frac{\cos(5x)}{\cos(-7x)}=\lim_{x\to\frac{\pi}{2}}\frac{5\sin(5x)}{-7\sin(-7x)}=\frac{-5}{7}$$

11) Evaluate $\lim\limits_{x\to\infty}\dfrac{5x^2-13x-6}{x+3}$ using L'Hopital's Rule if necessary.

Answer: Direct substitution gives you $\frac{\infty}{\infty}$. Keeping in mind that you can also use L'Hopital's Rule for $\frac{\infty}{\infty}$, using the rule once gives you

$$\lim_{x\to\infty}\frac{5x^2-13x-6}{x+3}=\lim_{x\to\infty}\frac{10x-13}{1}=+\infty$$

12) Describe the type of indeterminate form, if any, that is obtained with direct substitution for $\lim\limits_{x\to 0^+}(\sin x)^x$ and then evaluate the limit using L'Hopital's Rule if necessary.

Answers: Direct substitution on $\lim\limits_{x\to 0^+}(\sin x)^x$ yields 0^0 which, while it is an indeterminate form, it is not a quotient and therefore L'Hopital's Rule cannot be used. We must therefore try to turn the function into a quotient. When the variable is an exponent, it is usually a good idea to somehow manage to go to log-land. We can accomplish this by setting the entire limit equal to y and then taking the natural log of both sides. This gives us $\ln y = \ln \lim_{x \to 0^+} (\sin x)^x$. Since the log of a limit is equal to the limit of the log, you can manipulate this equation to yield $\ln y = \lim_{x \to 0^+} \ln(\sin x)^x$. The rules of logs allows us to rewrite this equation as $\ln y = \lim_{x \to 0^+} x \cdot \ln(\sin x)$. This allows us to finally write the limit as a quotient $\ln y = \lim_{x \to 0^+} \frac{\ln(\sin x)}{x^{-1}}$. Now we can finally apply L'Hopital's Rule. It will take 2 applications of L'Hopital's Rule in order to not get the indeterminate form $\frac{0}{0}$.

First application: $\ln y = \lim_{x \to 0^+} \frac{\ln(\sin x)}{x^{-1}} \to \lim_{x \to 0^+} \frac{\cos x}{-1x^{-2} \cdot \sin x}$. Now we can apply several basic manipulations to get $\ln y = \lim_{x \to 0^+} \frac{\cos x}{-1x^{-2} \cdot \sin x} \to \lim_{x \to 0^+} \frac{-x^2}{\tan x}$. Applying L'Hopital's Rule one more time yields $\ln y = \lim_{x \to 0^+} \frac{-x^2}{\tan x} \to \lim_{x \to 0^+} \frac{-2x}{\sec^2 x}$. Since $\lim_{x \to 0^+} \frac{-2x}{\sec^2 x} = \frac{0}{1}$, then $\ln y = 0$. Rewriting this log equation as an exponential gives us $e^0 = y$, which means that $y = 1$. Recalling that we originally made $y = \lim_{x \to 0^+} (\sin x)^x$ means that

$$\lim_{x \to 0^+} (\sin x)^x = 1$$