Teaching Notes for Calculus
Homework #11
Derivatives of Inverse Trig Functions and Relative Extrema

Derivatives of Inverse Trig Functions: Inverse trig functions can be written in 2 ways, $\arctan x \to \tan^{-1} x$. Both of these notations are equivalent and, when written as a function, results in a third form as follows:

$$f(x) = \arcsin x \to y = \sin^{-1} x \to \sin y = x$$

One of the goals of this lesson is to understand how to do a derivative on an inverse trig function and then to apply the extra derivative rules that come from these inverse trig functions.

For example: $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$. This is definitely not an intuitive result, so let's try to prove it!

Since $f'(x) = \frac{d}{dx}\arcsin x \to \frac{dy}{dx} = \frac{d}{dx}\sin^{-1} x \to \frac{d}{dx}\sin y = \frac{d}{dx}x$. Implicit differentiation yields the following equation: $\cos y \frac{dy}{dx} = 1\frac{dx}{dx} \to \frac{dy}{dx} = \frac{1}{\cos y}$. Therefore, $f'(x) = \frac{1}{\cos y}$. The issue now is that you have a function of x that's in terms of y. To fix that, you need the trig version of the Pythagorean Theorem, $\sin^2 y + \cos^2 y = 1$, and manipulate this to get $\cos y$ alone. This gives you $\cos y = \pm\sqrt{1-\sin^2 y}$. However, you must remember that the inverse of any function is the reflection of that function over the line y = x. In order for the inverse to also be a function, it has to pass the vertical line test. The graph of the inverse of sine fails the vertical line test an infinite number of times. Therefore, the only way to even start this problem is by restricting the domain of the sine function so, when you reflect it over y = x, the inverse passes the vertical line test. If you restrict the domain of the sine function to the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, it's inverse will also be a function. If this isn't clear to the students, make a quick graph of sine and prove it graphically. Once you have established this domain, it must be applied to the rest of the problem. That means that the cosine of any angle, on the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is always positive. Therefore, $\cos y = \pm\sqrt{1-\sin^2 y}$ can only be $\cos y = \sqrt{1-\sin^2 y}$.

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Remembering that $\sin y = x$ and that squaring both sides would give you $\sin^2 y = x^2$.

Substitution gives you $\cos y = \sqrt{1-x^2}$. Since $f'(x) = \dfrac{1}{\cos y}$, substitution yields the final answer $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$, where $x \neq \pm 1$.

Even though the following derivatives should be on their calculus sheet, you should write the derivatives of the other inverse trig functions on the board, listed below, and state that they all can be derived in a similar way:

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\operatorname{arccot} x = \frac{-1}{1+x^2} \qquad \frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\operatorname{arccsc} x = \frac{-1}{|x|\sqrt{x^2-1}}$$

EXTREMA

Relative extrema are all of the “valleys” and “peaks” of a graph on a given interval.

Absolute extrema are the lowest “valley” and highest “peak” on a given interval.

The Extreme Value Theorem – If a function is continuous on a closed interval then the function has both a minimum and maximum on the interval

For a function, extrema can only occur at:

1) Either endpoint of a closed interval
2) Any point on the interval where the derivative of the function equals zero
3) At any point on the interval where the derivative does not exist.

*Note - If any endpoint of an interval is open, then extrema cannot exist at that open end.

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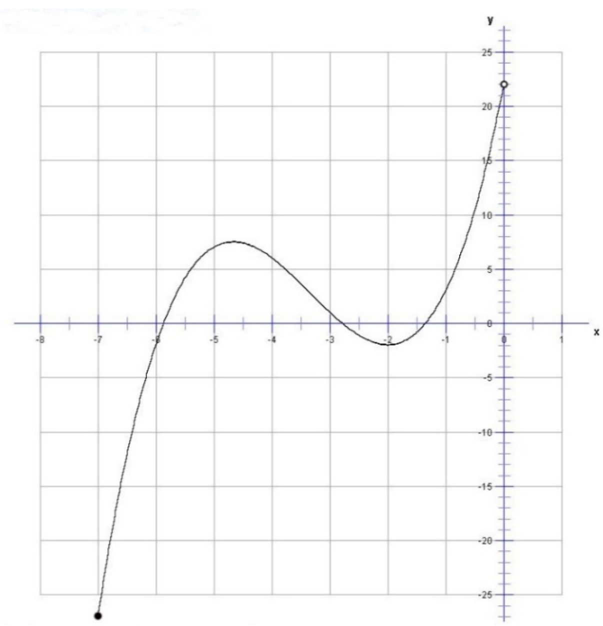
Classroom Examples:

1) Find the derivative of $g(x) = -5\arccos\frac{x}{3}$

Answer: Since this function involves arccos, we will need $\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$. As always, you must take the derivative of anything inside the main function you are differentiating. Therefore, in this problem, the answer, after substituting $\frac{x}{3}$ for x, gives you $g'(x) = \frac{5}{3\sqrt{1-\left(\frac{x}{3}\right)^2}}$. This answer can be manipulated to yield a much "prettier" answer by squaring and doing a jealousy game under the monster. This gives you $g'(x) = \frac{5}{3\sqrt{\frac{9-x^2}{9}}}$ which becomes $g'(x) = \frac{5}{\frac{3}{3}\sqrt{9-x^2}}$ and finally $g'(x) = \frac{5}{\sqrt{9-x^2}}$

2) Locate the absolute extrema of $f(x) = x^3 + 10x^2 + 28x + 22$ on the interval [-7,0)



Answers: First, graph the function and find all of the relative extrema on the interval. Keep in mind that, on any open end of an interval, there cannot be any type of extrema. Therefore, you can't find any extrema at $x = 0$, but you can find a relative extrema at $x = -7$. Substituting –7 into the function gives you a relative extrema at (–7, –27). There are also relative extrema at any point within the interval whenever the derivative of the function equals 0. The derivative is $f'(x) = 3x^2 + 20x + 28$. Setting this derivative equal to zero gives you

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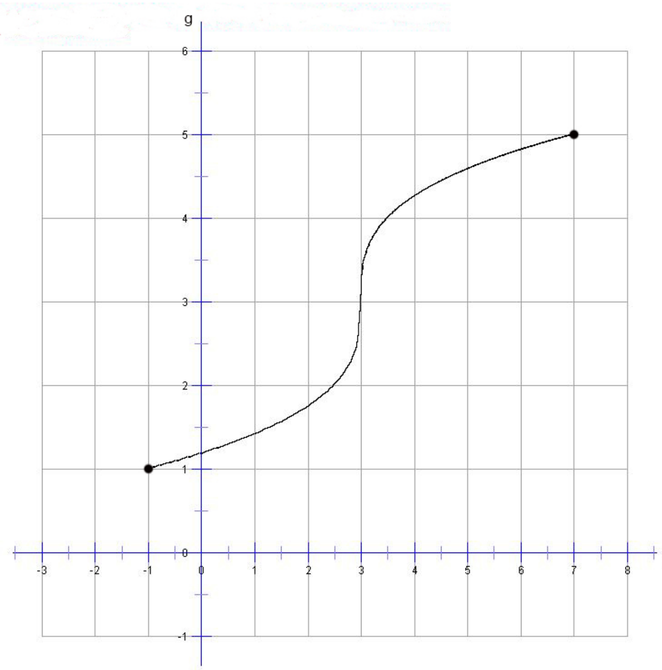
$0 = 3x^2 + 20x + 27$. Factoring and solving gives the answers of $x = \frac{-14}{3}$ and $x = -2$.

Substituting $x = \frac{-14}{3}$ into the original function gives you a relative extrema at $\left(\frac{-14}{3}, \frac{7474}{999}\right)$. Substituting $x = -2$ into the original function gives you a relative extrema at $(-2, -2)$. The absolute extrema on this interval are the points where the y value is greatest and smallest. Therefore, the absolute extrema are (–7, –27) and $\left(\frac{-14}{3}, \frac{7474}{999}\right)$.

3) Locate the absolute extrema of $g(x) = \sqrt[3]{2x-6} + 3$ on the interval [-1,7]



Answers: First, graph the function and find all of the relative extrema on the interval. Keep in mind that, on a closed interval, both endpoints are extrema. Therefore, you can find an extrema at both $x = -1$, and at $x = 7$. Substituting –1 and 7 into the function gives you relative extrema at (–1, 1) and at (7, 5). There are also relative extrema at any point within the interval whenever the derivative of the function equals 0. The derivative is $g'(x) = \frac{2}{3\sqrt[3]{(2x-6)^2}}$. Setting this derivative equal to zero gives you $0 = \frac{2}{3\sqrt[3]{(2x-6)^2}}$.

Clearing fractions ends up eliminating x from the entire equation. Therefore, there are no values of x that will make the derivative 0. Therefore the absolute extrema must be at the endpoints. The answers are (–1, 1) and at (7, 5).

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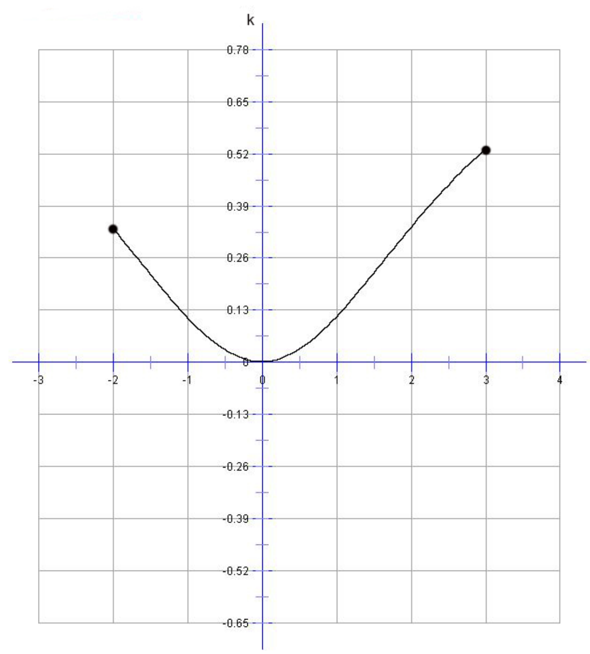
4) Find the derivative of $f(x) = \arctan\frac{x}{m^3}$

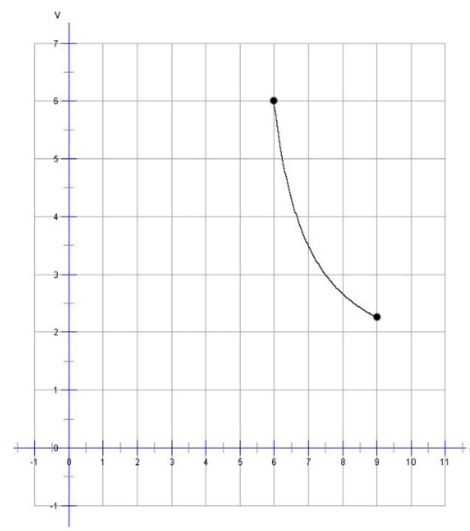
Answer: Since this function involves arctan, we will need $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$. As always, you must take the derivative of anything inside the main function you are differentiating. Therefore, in this problem, the answer, after substituting $\frac{x}{m^3}$ for x, gives you $f'(x) = \frac{1}{m^3\left(1+\left(\frac{x}{m^3}\right)^2\right)}$. This answer can be manipulated to yield a much "prettier" answer by squaring and doing a jealousy game on the bottom. This gives you $f'(x) = \frac{1}{m^3\left(\frac{m^6+x^2}{m^6}\right)}$ which finally simplifies to $f'(x) = \frac{m^3}{m^6+x^2}$.

5) Locate the absolute extrema of $k(x) = \frac{x^2}{x^2+8}$ on the interval [-2,3]



Answers: First, graph the function and find all of the relative extrema on the interval. Keep in mind that, on a closed interval, both endpoints are extrema. Therefore, you can find an extrema at both $x = -2$, and at $x = 3$. Substituting –2 and 3 into the function gives you relative extrema at $\left(-2, \frac{1}{3}\right)$ and at $\left(3, \frac{9}{17}\right)$ There are also relative extrema at any point within the interval whenever the derivative of the function equals 0. The derivative is $k'(x) = \frac{-2x^3}{\left(x^2+8\right)^2} + \frac{2x}{x^2+8}$. Setting this derivative equal to zero gives you $\dfrac{-2x^3}{\left(x^2+8\right)^2}=\dfrac{2x}{x^2+8}$. Clearing fractions gives you $-2x^3=2x\left(x^2+8\right)$. Multiplying and making it equal to zero yields $4x^3+16x=0$. Factoring and solving yields $x=0,2i,-2i$. The only one of these three answers that are within the interval is 0. Substituting $x=0$ into the original function gives you a relative extrema at (0, 0). Therefore, the absolute extrema on this interval are (0, 0) and $\left(3,\dfrac{9}{17}\right)$.

6) Locate the absolute extrema of $v(t)=\dfrac{t}{t-5}$ on the interval [6,9]



Answers: First, graph the function and find all of the relative extrema on the interval. Keep in mind that, on a closed interval, both endpoints are extrema. Therefore, you can find an extrema at both $x=6$, and at $x=9$. Substituting 6 and 9 into the function gives you relative extrema at (6, 6) and at $\left(9,\dfrac{9}{4}\right)$ There are also relative extrema at any point within the interval whenever the derivative of the function equals 0. The derivative is $v'(t)=\dfrac{-t}{(t-5)^2}+\dfrac{1}{(t-5)}$. Setting this derivative equal to zero and clearing fractions gives you $0=-t+t-5\rightarrow 0=-5$, which means there are no answers for t. Therefore the absolute extrema are (6, 6) and $\left(9,\dfrac{9}{4}\right)$.

7) Find the derivative of $g(x)=\dfrac{\arcsin(4x)}{3x}$

Answer: Since this function involves arcsin, we will need $\dfrac{d}{dx}\arcsin x=\dfrac{1}{\sqrt{1-x^2}}$ and either the product or quotient rule. As always, you must take the derivative of anything inside the main function you are differentiating. Therefore, the answer, after substituting

$4x$ for x and using the product rule, gives you $g'(x) = \dfrac{4}{3x\sqrt{1-(4x)^2}} + \dfrac{-3\sin^{-1}(4x)}{9x^2}$.

Which simplifies to $g'(x) = \dfrac{4}{3x\sqrt{1-16x^2}} - \dfrac{\sin^{-1}(4x)}{3x^2}$.