Teaching Notes for Calculus
Homework #12
Critical Numbers, Rolle's & The Mean Value Theorem, and Concavity

If $f'(c)=0$ and the function is defined at c, then c is considered to be a **CRITICAL NUMBER** This means something special is happening at c! *While restrictions that produce vertical asymptotes are not technically considered to be critical numbers, they must be identified and considered as "special Critical Number like places on a graph."*

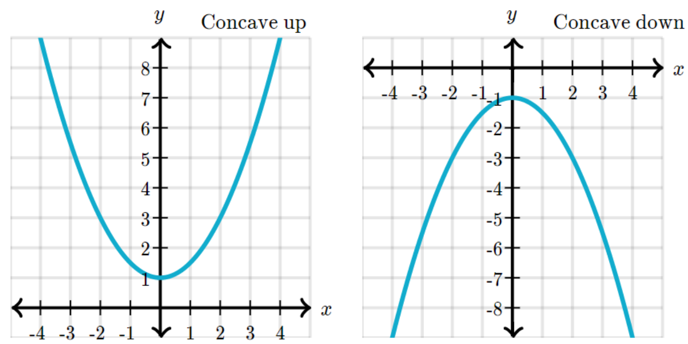
Rolle's Theorem – If f(x) is both continuous and differentiable on the interval [a,b] and f(a) = f(b), then there is at least one number c on (a,b) such that $f'(c)=0$. *Draw a graph where f(a) = f(b) and then try to connect the two points with ANY function…this will provide visual "proof" that there is at least one number c on (a,b) such that $f'(c)=0$.

The Mean Value Theorem – If f(x) is both continuous and differentiable on the interval (a,b), then there exists a number c on (a,b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$ *Not very practical in problem solving but considered to be one of the most important theorems in calculus because it is used to prove many important concepts* *Explain that $\frac{f(b)-f(a)}{b-a}$ is basically the "average" slope of the function on the interval as measured from one end to the other…and it is a line connecting those two points! Remember that $f'(c)$ is the slope at any point on the interval (a,b) and that the overall derivative function, $f'(x)$, is some sort of line or curve connecting f(a) to f(b). Draw a picture to "prove" that it is impossible to draw any line or curve for $f'(x)$, between f(a) to f(b), that doesn't cross the straight line you made from the average or "mean" slope, $\frac{f(b)-f(a)}{b-a}$. Any intersections between those two lines are the c's that must exist on (a, b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$. *

The First Derivative Test – If $f'(x)=0$ for all x on (a, b) then the function is constant or "flat" (it's a bunny slope graph) on the interval. If $f'(x)>0$ for all x on (a, b) then the function is increasing (consistently going up as you travel from left to right) on the interval. If $f'(x)<0$ for all x on (a, b) then the function is decreasing (consistently going down as you travel from left to right) on the interval.

If $f'(x)$ goes from negative just before c to positive right after c, then c is a relative minimum. If $f'(x)$ goes from positive just before c to negative right after c, then c is a relative maximum. *This good to know, but it is A LOT of work to determine minimums and maximums this way…thankfully, there is a better way to determine this!!!*

Concavity and the second derivative test: If f(x) is differentiable on an open interval, I, and $f''(x) > 0$ for all x on I, then the graph is concave upwards on that interval. If f(x) is differentiable on an open interval, I, and $f''(x) < 0$ for all x on I, then the graph is concave downwards on that interval.



Points of Inflection occur when there is a change in concavity. Points of inflection only occur when $f''(x) = 0$ or could occur when $f''(x)$ is undefined but only if the concavity switches at that point AND the original function is continuous at that point.

This means that inflection points cannot exist where the graph of the original function has a discontinuity or where it creates vertical asymptotes, but there could be a change in concavity at these at these asymptotes.

There is an easier way to determine minimums* If $f'(c) = 0$ and $f''(c) > 0$ then $f(c)$ is a relative minimum

There is an easier way to determine maximums* If $f'(c) = 0$ and $f''(c) < 0$ then $f(c)$ is a relative maximum

*Note – If you are trying to find minimums and maximums the "easier way" and $f''(c) = 0$, then the second derivative test fails and you must use the first derivative test (the one with A LOT of work)!

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Classroom Examples

1) Determine whether Rolle's Theorem can be applied to $f(x)=\dfrac{x^2-9}{2x}$ on the interval [-3,3]. If it can be applied, then find all values of c, on the interval [-3,3] such that $f'(c)=0$.

Answers: Since $f(x)$ is undefined at $x=0$, and zero is inside the interval [-3,3], the function is discontinuous on the interval, therefore Rolle's Theorem cannot be applied.

2) Determine whether Rolle's Theorem can be applied to $f(x)=x^2-3x-10$ on the interval [5,-2]. If it can be applied, then find all values of c, on the interval [5,-2] such that $f'(c)=0$.

Answers: $f(x)=x^2-3x-10$ is a polynomial. Therefore, it is both continuous and differentiable on the interval [5,-2]. Unfortunately, there is one more condition which has to be met in order for Rolle's Theorem to apply, $f(5)$ must equal $f(-2)$. Since $f(5)=0$ and $f(-2)=0$, Rolle's Theorem does apply! Therefore, since $f'(x)=2x-3$ and Rolle's Theorem applies, there MUST be at least one x value, c, on [5,-2] where $f'(c)=0$. Substitution yields $0=2c-3$ which means there is one answer: $c=\dfrac{3}{2}$

3) Apply the Mean Value Theorem to $f(x)=x^2-5x+7$ on the interval $[-1,3]$. Find all values of c on the interval $[-1,3]$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Answers: Begin by finding $\dfrac{f(b)-f(a)}{b-a}$ where $a=-1$ and $b=3$. Substitution yields $\dfrac{f(3)-f(-1)}{(3)-(-1)}=-3$. Now find $f'(x)$. $f'(x)=2x-5$ and, since $f'(c)=-3$, then $f'(c)=-3=2x-5$. Solving yields just one answer: $c=1$

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4) Find the critical numbers of $f(x)=\frac{x-4}{x^2}$, find all vertical asymptotes, find the open intervals on which $f(x)$ is increasing or decreasing, and locate all relative extrema.

Answers: Critical Numbers arise when the derivative of the function equals zero, as long as the original function is defined at that number. Taking the derivative of the function yields $f'(x)=\frac{1}{x^2}-\frac{2x-8}{x^3}\rightarrow\frac{-1}{x^2}+\frac{8}{x^3}$. Setting the derivative equal to zero yields $\frac{-1}{x^2}+\frac{8}{x^3}=0$. Clearing fractions results in $-1x+8=0$. Solving, gives you one Critical Number of $x=8$. Keep in mind that, although it is not considered a Critical Number, something special also happens at $x=0$ as this is a vertical asymptote. A quick sketch of the function will help to verify your conclusions and will help to guide the rest of your answers. The best way, quickest way, to determine whether the critical number is relative minimum or maximum is to take the second derivative and evaluate it at 8. $f''(x)=\frac{2}{x^3}-\frac{24}{x^4}$. Therefore, $f''(8)=\frac{-1}{512}$. If the first derivative is zero and the second derivative is negative, then the critical number represents a relative maximum. Substituting 8 into $f(x)=\frac{x-4}{x^2}$ gives you $f(8)=\frac{1}{16}$. Therefore, there is a relative maximum at the point $\left(8,\frac{1}{16}\right)$. This also indicates that to the right of that point, until you hit another "special" number, the function will be decreasing and, to the left of the point, until you hit another "special" number, the function will be increasing. There are no special numbers to the right of the point, so the function is decreasing on the interval $(8,\infty)$. To the left, however, there is a "special" number. The function is undefined at zero. Therefore, the function must be increasing on the interval (0, 8). Since there are no other "special" numbers, the final interval will be $(-\infty,0)$ but, the important question is if it is increasing or decreasing over this interval. A quick examination of the sketch you made shows that the function is decreasing on that interval, but that is not an acceptable rationale. The proof of what appears to be self-evident is to evaluate the second derivate at some point within that interval and testing for concavity. Choosing a random number,

like –4 for x, and substituting it into the second derivative formula gives: $f''(-4) = \frac{-1}{8}$.
Since the second derivative is negative, then that entire interval is concave down, without having a relative maximum. Therefore, the interval $(-\infty, 0)$ is decreasing.

5) Without the use of a graphing calculator, graph, find all relative extrema, all points of inflection, all vertical asymptotes, and find the concavity on all intervals for $f(x) = \cos x - \sin x$ on the interval $0 \le x \le 2\pi$.

Answers: First, it would be helpful to find, especially for a trig function, the starting point, middle point, and end point on the closed interval. Substituting zero in for x gives you an answer of 1 or the point $(0,1)$. Substituting π in for x gives you an answer of –1 or the point $(\pi, -1)$. Substituting 2π in for x gives you an answer of 0 or the point $(2\pi, 0)$. Using graph paper, draw the x and y axes, and mark, exactly, all three points. Second, observe that, over the interval $[0, 2\pi]$, there are no values that will make the function undefined so there are no "special" vertical asymptotes. Next, find any Critical Numbers by taking the first derivative and setting it equal to zero.
$f'(x) = -\sin x - \cos x$ so we need to solve the equation $-\sin x - \cos x = 0$.
Manipulating this equation yields $\tan x = -1$. Solving for x gives the answers $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$. These are the critical numbers. Evaluate the second derivative at these two numbers will tell you if they are relative minimums, relative maximums, or points of inflection. The second derivative is $f''(x) = -\cos x + \sin x$, so

$f''\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$ and $f''\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$. Therefore, at $x = \frac{3\pi}{4}$, there is a relative minimum (concave up) and, at $x = \frac{7\pi}{4}$, there is a relative maximum (concave down). Substituting these two numbers into the original function will give you the actual points for the minimum and maximum. Therefore, the relative minimum is at

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$\left(\frac{3\pi}{4},-\sqrt{2}\right)$ and the relative maximum is at $\left(\frac{7\pi}{4},\sqrt{2}\right)$. Plot both of these points as accurately as possible. Points of inflection only occur when the second derivative is either zero or, under special circumstances, undefined. We already know that there are no values on the interval that make the addition or subtraction of sines and cosines undefined. Therefore, set the second derivative equal to zero and solve.

$f''(x) = -\cos x + \sin x = 0 \rightarrow \tan x = 1$. Solving for x gives you $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

Substituting these values into the original function gives you $\left(\frac{\pi}{4},0\right)$ and $\left(\frac{5\pi}{4},0\right)$ as the two points of inflection. Find both of these points exactly on your graph paper. The last thing that needs to be calculated, before graphing, are the intervals where the function is concave up and concave down. Remembering that concavity switches at points of inflection, and going from left to right on the graph, the graph must be concave down from the point $(0,1)$ to the point $\left(\frac{\pi}{4},0\right)$ or on the interval $\left(0,\frac{\pi}{4}\right)$. The graph must be concave up from the point $\left(\frac{\pi}{4},0\right)$, through the points $\left(\frac{3\pi}{4},-\sqrt{2}\right)$ and $(\pi,-1)$, to the point $\left(\frac{5\pi}{4},0\right)$ or on the interval $\left(\frac{\pi}{4},\frac{5\pi}{4}\right)$. The graph must be concave down from the point $\left(\frac{5\pi}{4},0\right)$, through the point $\left(\frac{7\pi}{4},\sqrt{2}\right)$, to the point $(2\pi,0)$ or on the interval $\left(\frac{5\pi}{4},2\pi\right)$. Finally, complete the graph, from left to right, with a nice, smooth curve.

6) The speed limit on a certain highway is 60 miles per hour. Two stationary police cars with radar detectors are 7 miles apart on a highway. As a car passes the first police car, its speed is clocked at 55 miles per hour. Six minutes later the car passes the second police car and its speed is clocked at 45mph. Prove that the car must have exceeded the speed limit at some point during the six minutes.

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Answer: Average Speed is the $\dfrac{\text{change in distance}}{\text{change in time}} = \dfrac{s(t_2)-s(t_1)}{t_2-t_1} = \dfrac{s\left(\frac{1}{10}\right)-s(0)}{\frac{1}{10}-0} = \dfrac{7-0}{\frac{1}{10}-0} = 70$. Therefore, according to the Mean Value Theorem, there must be at least one time, t, between 0 and $\frac{1}{10}$ hours (6 minutes), where the speed of the car is exactly 70 mph. This proves that the car must have exceeded the speed limit.

7) Find a function of the form $f(x) = ax^3 + bx^2 + cx + d$ if it has a relative minimum at (5,-1), a relative maximum at (3,3), and an inflection point at (4,1).

Answer: This can be a very tricky problem because of the inflection point. Most students attempt to find the second derivative and then use the 4 from the inflection point in order to create an equation that they will, later on in the problem, try to combine with other equations to solve a system of equations. The problem with that is that the inflection point equation only contains redundant information from the other equations. Trying to use it to solve the system will always result in an answer of All Real Numbers! The first thing to do, and hopefully the most obvious, is to create three equations from the original function using the three points that we know are actually on the function. Substituting the minimum and maximum points into the original function results in two equations: point (5,-1) results in $125a + 25b + 5c + d = -1$ and point (3,3) results in $27a + 9b + 3c + d = 3$. Substituting the inflection point gives you the third equation: point (4,1) results in $64a + 16b + 4c + d = 1$. Unfortunately, you now have 4 unknowns and only 3 equations. The fourth equation can come from using either the minimum point or maximum point in the first derivative equation and setting it equal to zero. The first derivative would be $f'(x) = 3ax^2 + 2bx + c$. Using the maximum point, so x is 3 and the derivative is 0, gives you the fourth equation: $27a + 6b + c = 0$. You now have a 4 by 4 system of equations which you can solve using any of the methods learned in Algebra II or Pre-Calculus. When you solve the 4x4, you will get $a = 1, b = -12, c = 45,$ and $d = 0$ so the original function was $f(x) = 1x^3 - 12x^2 + 45x + 0$.