# Teaching Notes for Calculus
# Homework #13
# Curve Sketching Without a Calculator and Optimization Problems

**Guidelines for Curve Sketching without a graphing calculator:**

1) Determine the domain and range of the function

2) Determine the intercepts and asymptotes

3) Find all values of $f'(x)$ and $f''(x)$ that equal 0 or are undefined to locate, relative extrema, inflection points.

4) Use the second derivative on all intervals to determine concavity – If the second derivative is positive, then it is concave up on the interval…negative means concave down…

Guidelines for Solving Optimization Problems:

1) Make a sketch and label all important information in the sketch

2) Write a general equation that describes the problem and determine all domains

3) Try to manipulate that equation to reduce it to a single variable

4) Use derivatives to find maximums or minimums

## Classroom Examples

1) Without the use of a graphing calculator, sketch the graph of $y = 2x^4 - 18x^2$ and be sure to show all relative extrema, points of inflection, intercepts, or asymptotes on the graph and determine the concavity.

Answer: Follow these 4 steps:

1) Determine the domain and range of the function – Since the function is a polynomial, there are no restrictions and the graph is continuous. Therefore, the domain would be All Real Numbers. The function is Even so it will follow the shape of a parabola…the leading coefficient is positive so the end behavior is up on both the left and right…

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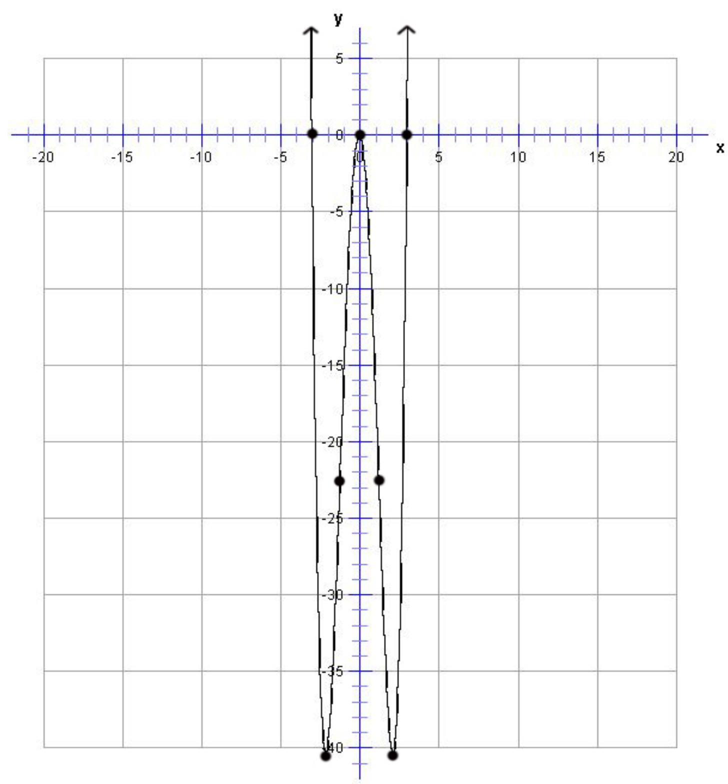
2) Determine the intercepts and asymptotes – Try to factor the polynomial…this gives you $y = 2x^2(x+3)(x-3)$. Therefore, the x intercepts are $(-3,0),(3,0),(0,0)$ and, since $(0,0)$ is a double root, the graph just touches the x axis at 0. The y intercept would be $(0,0)$. Since there are no restrictions, there are no asymptotes.

3) Find all values of $f'(x)$ and $f''(x)$ that equal 0 or are undefined to locate, relative extrema, inflection points – Setting $f'(x) = 8x^3 - 36x$ equal to zero and factoring gives $x = 0, \frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}$ when the slope is 0. This means that that there are relative extrema at $x = 0, \frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}$ which gives you the points $(-\frac{3\sqrt{2}}{2}, \frac{-81}{2}), (\frac{3\sqrt{2}}{2}, \frac{-81}{2}), (0,0)$ Setting $f''(x) = 24x^2 - 36$ equal to 0 and factoring tells you that there are inflections at $x = \frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{2}$ which give you inflection points at $(-\frac{\sqrt{6}}{2}, \frac{-45}{2}), (\frac{\sqrt{6}}{2}, \frac{-45}{2})$.

4) Use the second derivative on all intervals to determine concavity – For concavity, the intervals are determined by vertical asymptotes and inflection points. Since there are no asymptotes, only the two inflection points break up the graph into these three intervals… $x < -\frac{\sqrt{6}}{2}$, $-\frac{\sqrt{6}}{2} < x < \frac{\sqrt{6}}{2}$, and $x > \frac{\sqrt{6}}{2}$.

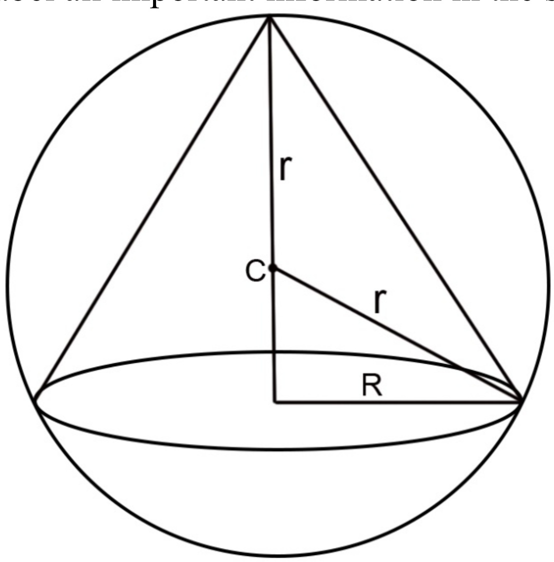



Keep in mind that you can choose ANY point in an interval to test the entire interval. Therefore, in most cases, you can choose integers to test the sections which will make the math easier. Choosing $x = -4$ and substituting it into the second derivative yields $f''(-4) = 24(-4)^2 - 36 = 348$ which is positive. This means that the graph is concave up in the left interval. Choosing $x = 0$ and substituting it into the second derivative yields $f''(0) = 24(0)^2 - 36 = -36$ which is negative. This means that the graph is concave down in the middle interval. Choosing $x = 3$ and substituting it into the second derivative yields $f''(3) = 24(3)^2 - 36 = 180$ which is positive. This means that the graph is concave up in the right interval. You now have enough information to make a very accurate sketch of this graph:

2) Find the volume of the largest right circular cone of radius R that can be inscribed in a sphere of radius r, with center C.

Answer: Follow these 4 steps:

1) Make a sketch and label all important information in the sketch:




2) Write a general equation that describes the problem and determine all domains:

Since we are describing the physical dimensions of real objects, all domains are any real number greater than zero. The question asks us to maximize the volume of the cone so we should start with the formula, $V = \dfrac{\pi R^2 h}{3}$. Take care, in this problem, to differentiate and correctly use, r, the radius of the sphere, and R, the radius of the base of the cone. Also, keep in mind that, while the general answer to this question must vary with the radius of the sphere, the shape of the inscribed cone must be independent of r. In other words, the ratio of the height of the cone to its radius will remain constant, regardless of the radius of the sphere. Therefore, the radius of the sphere, r, can be considered to be a constant for the purposes of the differentiation.

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3) Try to manipulate that equation to reduce it to a single variable:

In this case, try to write $V = \dfrac{\pi R^2 h}{3}$ in terms of only R, the radius of the cone, while eliminating the height, h. Keep in mind, however, that, as you do this manipulation, it is fine if the radius of the sphere, r, enters the volume equation as it is considered a constant, as mentioned in Step 2. The first step would be to find the height of the cone, h, in terms of r. Using the smaller of the two right triangles in the diagram, calling the missing leg, x, and applying the Pythagorean Theorem yields $x^2 + R^2 = r^2 \rightarrow x = \sqrt{r^2 - R^2}$. The diagram reveals that $h = r + x$. Substitution, therefore, yields $h = r + \sqrt{r^2 - R^2}$. Substituting this new equation into the volume equation yields $V = \dfrac{\pi R^2\left(r + \sqrt{r^2 - R^2}\right)}{3}$, which can be manipulated into the equation, $V = \dfrac{\pi r R^2}{3} + \dfrac{\pi R^2 \sqrt{r^2 - R^2}}{3}$.

4) Use derivatives to find maximums or minimums:

Since we are trying to maximize the volume of the cone, V, as it relates to the radius of the cone, R, we should take the derivative of V with respect to R while remembering that r is considered a constant for this derivative. The derivative yields the equation $\dfrac{dV}{dR} = \dfrac{2\pi r R}{3} + \dfrac{2\pi R\sqrt{r^2 - R^2}}{3} - \dfrac{\pi R^3}{3\sqrt{r^2 - R^2}}$. The volume of the cone, V, would be maximized when $\dfrac{dV}{dR} = 0$. Therefore, solve

$\dfrac{2\pi r R}{3} + \dfrac{2\pi R\sqrt{r^2 - R^2}}{3} - \dfrac{\pi R^3}{3\sqrt{r^2 - R^2}} = 0$. Clearing fractions yields

$2\pi r R\sqrt{r^2 - R^2} + 2\pi R\left(r^2 - R^2\right) - \pi R^3 = 0$. Simplifying this equation yields

$2\pi r R\sqrt{r^2 - R^2} + 2\pi r^2 R - 3\pi R^3 = 0$. Factoring results in equation

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$\pi R\left(2r\sqrt{r^2-R^2}+2r^2-3R^2\right)=0$. This means that either

$2r\sqrt{r^2-R^2}+2r^2-3R^2=0$ or $\pi R=0$. As stated previously, $R\neq 0$. Therefore, we must try to solve the remaining equation, $2r\sqrt{r^2-R^2}+2r^2-3R^2=0$. To solve this equation we must isolate the radical and square both sides. Therefore,

$2r\sqrt{r^2-R^2}=3R^2-2r^2 \rightarrow 4r^2\left(r^2-R^2\right)=\left(3R^2-2r^2\right)^2 \rightarrow$
$4r^4-4r^2R^2=+4r^4 \rightarrow 0=9R^4-8r^2R^2 \rightarrow$
$0=R^2\left(9R^2-8r^2\right)\rightarrow 0=R^2\left(3R+2\sqrt{2}r\right)\left(3R-2\sqrt{2}r\right)$. This equation yields four answer for R: $R=0, R=0, R=\frac{-2\sqrt{2}r}{3}, R=\frac{2\sqrt{2}r}{3}$. Since R can't be less than or equal to 0, only one answer remains, $R=\frac{2\sqrt{2}r}{3}$. So, for a sphere of radius r, the dimensions of the inscribed cone are $R=\frac{2\sqrt{2}r}{3}$ and $h=r+\sqrt{r^2-\frac{8r^2}{9}}\rightarrow h=r+\sqrt{\frac{r^2}{9}}\rightarrow h=\frac{4r}{3}$. Therefore, the maximum volume of such a cone inscribed in a sphere of radius, r, would be expressed as
$V=\frac{32\pi r^3}{81}$ .

3) Without the use of a graphing calculator, sketch the graph of $f(x)=\frac{-x^3}{x^2-4}$ and be sure to show all relative extrema, points of inflection, intercepts, or asymptotes on the graph and determine the concavity.

Answer: Follow these 4 steps:

1) Determine the domain and range of the function – There are two restrictions for this function, $x\neq 2,-2$. Therefore, the domain would be All Real Numbers except –2 and 2. The graph is discontinuous due to the restrictions. At this point, the range of this function is not obvious so we will postpone finding the range and move on to the other questions for now.

2) Determine the intercepts and asymptotes – The x and y intercepts are the same and occur at the point (0, 0). Both of the restrictions lead to vertical asymptotes, one at

$x=-2$ and the other at $x=2$. There are no horizontal asymptotes but there is an oblique asymptote since the degree of the top is one greater than the degree on the bottom. Dividing the bottom into the top yields the oblique asymptote $y=-x$. Both the vertical and oblique asymptotes will prove indispensable when making the graph as the function will approach and by contained by theses lines.

3) Find all values of $f'(x)$ and $f''(x)$ that equal 0 or are undefined to locate, relative extrema, inflection points – Find the first derivative using either the quotient or product rule. Using the product rule, $f'(x)=\frac{2x^4}{\left(x^2-4\right)^2}+\frac{-3x^4+12x^2}{\left(x^2-4\right)^2}\rightarrow\frac{-x^4+12x^2}{\left(x^2-4\right)^2}$. Setting $f'(x)$ equal to zero and clearing fractions yields $-x^4+12x^2=0$. Factoring results in $-x^2\left(x^2-12\right)=0$ which yields two possible solution equations, either $x^2=0$ or $x^2-12=0$. When $x^2=0$, then $x=0$. Solving the other equation yields $x=\pm\sqrt{12}\rightarrow x=\pm2\sqrt{3}$. We therefore have three relative extrema. One extrema is located at $(0,0)$, another at $\left(2\sqrt{3},-3\sqrt{3}\right)$ and the last is located at $\left(-2\sqrt{3},3\sqrt{3}\right)$. We now find the second derivative by either using the quotient rule of product rule. Using the product rule gives you $f''(x)=\frac{4x^5-48x^3}{\left(x^2-4\right)^3}+\frac{-4x^5+24x^3+16x^3-96x}{\left(x^2-4\right)^3}\rightarrow\frac{-8x^3-96x}{\left(x^2-4\right)^3}$.

Remember that all real solutions AND restrictions of the second derivative are places where inflections occur. Therefore, an inflection occurs at $x=-2$ and $x=2$. Setting the second derivative equal to 0 and clearing fractions yields the equation $-8x^3-96x=0$. Factoring yields $-8x(x^2+12)=0$Therefore, there is another inflection at $x=0$, which gives you an inflection point at $(0,0)$.

4) Use the second derivative on all intervals to determine concavity – For concavity, the intervals are determined by vertical asymptotes and inflection points. Since there are two vertical asymptotes and one inflection point, this graph is broken up into 4 continuity intervals… $x<-2,-2<x<0,0<x<2$ and $x>2$. Keep in mind that you can choose ANY point in an interval to test the entire interval. Therefore, in most cases, you can choose integers to test the sections which will make the math easier. Choosing $x=-3$ and substituting it into the second derivative yields $f''(-3)=\frac{-8(-3)^3-96(-3)}{\left((-3)^2-4\right)^3}=\frac{504}{125}$, which is

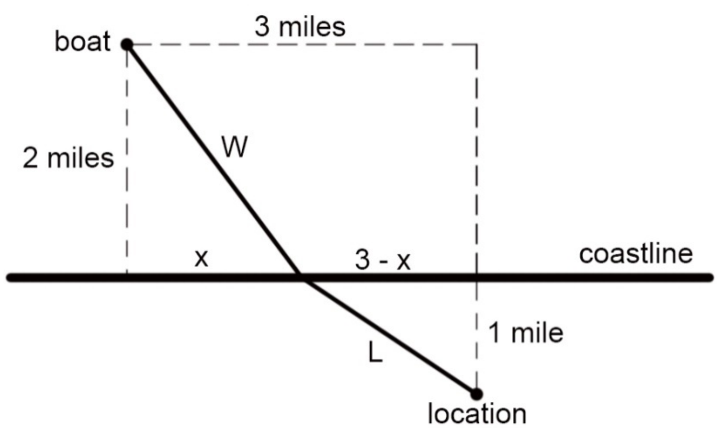
positive. This means that the graph is concave up in the 1st interval. Choosing $x=-1$ and substituting it into the second derivative yields $f''(-1)=\dfrac{-8(-1)^3-96(-1)}{\left((-1)^2-4\right)^3}=\dfrac{104}{-27}$, which is negative. This means that the graph is concave down in the 2nd interval. Choosing $x=1$ and substituting it into the second derivative yields $f''(1)=\dfrac{-8(1)^3-96(1)}{\left((1)^2-4\right)^3}=\dfrac{104}{27}$, which is positive. This means that the graph is concave up in the 3rd interval. Finally, Choose $x=3$ and substituting it into the second derivative yields $f''(3)=\dfrac{-8(3)^3-96(3)}{\left((3)^2-4\right)^3}=\dfrac{-504}{125}$, which is negative. This means that the graph is concave down in the 4th interval. You now have enough information to make a very accurate sketch of this graph and determine the nature of the extrema and the range:

From this graph, it is obvious that the range is All Real Numbers. It is also clear that the point $\left(-2\sqrt{3},3\sqrt{3}\right)$ is a relative minima, point $\left(2\sqrt{3},-3\sqrt{3}\right)$ is a relative maxima, and the point $(0,0)$ is just a critical point of inflection.

4) A man is in a boat 2 miles from the nearest point on the coast. He is to go to a location 3 miles down the coast and 1 mile inland. Assuming that the coastline is a straight line and that he can row at a rate of 4 miles per hour and walk at a rate of 4 miles per hour, towards what point on the coastline should he row in order to reach his location in the least amount of time and how long will the entire trip take?

Answer: Follow these 4 steps:

1) Make a sketch and label all important information in the sketch:



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2) Write a general equation that describes the problem and determine all domains:

Since we are describing the physical dimensions of real objects, all domains are any real number greater than zero. The question asks us to minimize the total time of the trip so we should start with and applicable formulas for the distances, times, and rates involved in the trip. The main equation, the one that must be minimized, is the formula for the total time of the trip, $T = T_W + T_L$. The trip is comprised of two lengths, both of which are the hypotenuses of two different right triangles. Therefore, for the boat segment of the trip, and calling the distance, y, that the boat is from the coastline, the Pythagorean Theorem yields $x^2 + y^2 = W^2$. For the land segment of the trip, and calling the distance, s, that the final location is from the coastline, the Pythagorean Theorem yields $(3-x)^2 + s^2 = L^2$. There are two other formulas you can write: $W = R_W T_W$ for the boat part of the trip and $L = R_L T_L$ for the land part of the trip.

3) Try to manipulate the main equation to reduce it to a single variable:

In this case, try to write $T = T_W + T_L$ in terms of only x, the distance down the coastline where where the boat lands. Substitution yields

$T = \frac{W}{R_W} + \frac{L}{R_L} \rightarrow T = \frac{\sqrt{x^2 + y^2}}{R_W} + \frac{\sqrt{(3-x)^2 + s^2}}{R_L}$. Now take stock of what we know:

$s = 1, y = 2, R_L = \frac{dL}{dt} = 4, R_W = \frac{dW}{dt} = 4$. Substitution yields

$$T = \frac{\sqrt{x^2 + (2)^2}}{4} + \frac{\sqrt{(3-x)^2 + (1)^2}}{4} \rightarrow T = \frac{\sqrt{x^2 + 4}}{4} + \frac{\sqrt{x^2 - 6x + 10}}{4}.$$

4) Use derivatives to find maximums or minimums:

Since we are trying to minimize the total time of the trip, T, as it relates to the distance down the coastline where the boat lands, x, we should take the derivative of T with respect to x. Differentiating yields $\frac{dT}{dx} = \frac{x}{4\sqrt{x^2+4}} + \frac{x-3}{4\sqrt{x^2-6x+10}}$. The total time, T, would be minimized when $\frac{dT}{dx} = 0$. Therefore, solve $\frac{x}{4\sqrt{x^2+4}} + \frac{x-3}{4\sqrt{x^2-6x+10}} = 0$. Clearing fractions yields $x\sqrt{x^2-6x+10} + (x-3)\sqrt{x^2+4} = 0$. Simplifying this equation involves isolating and eliminating each radical according to the following process:

$$x\sqrt{x^2-6x+10} = -(x-3)\sqrt{x^2+4} \rightarrow x^2(x^2-6x+10) = (x-3)^2(x^2+4) \rightarrow$$
$$x^4 - 6x^3 + 10x^2 = x^4 - 6x^3 + 13x^2 - 24x + 36 \rightarrow 3x^2 - 24x + 36 = 0.$$ Factoring yields $3(x-2)(x-6) = 0$. This means that either $x = 2$ or $x = 6$. Since x cannot equal 6 because the upper limit on the length of x is 3, $x = 2$ is the only answer for x. Therefore, the boat must land exactly 2 miles down the coastline in order to minimize the total time of the trip. As previously determined, $T = \frac{\sqrt{x^2+4}}{4} + \frac{\sqrt{x^2-6x+10}}{4}$. Therefore, if $x = 2$, the minimum time of the entire trip would be $T = \frac{\sqrt{(2)^2+4}}{4} + \frac{\sqrt{(2)^2-6(2)+10}}{4}$ which simplifies to exactly $\frac{3\sqrt{2}}{4}$ hours (approximately 1.06 hours).

5) Without the use of a graphing calculator, sketch the graph of $f(x) = \frac{-5x}{\sqrt{x^2+16}}$ and be sure to show all relative extrema, points of inflection, intercepts, or asymptotes on the graph and determine the concavity.

Answer: Follow these 4 steps:

1) Determine the domain and range of the function – There are no restrictions for this function as there is a radical on the bottom whose value never goes below 4, regardless of the value of x. Therefore, the domain would be All Real Numbers. The graph is also continuous for the same reasons. The range, while not as obvious as the domain, is also

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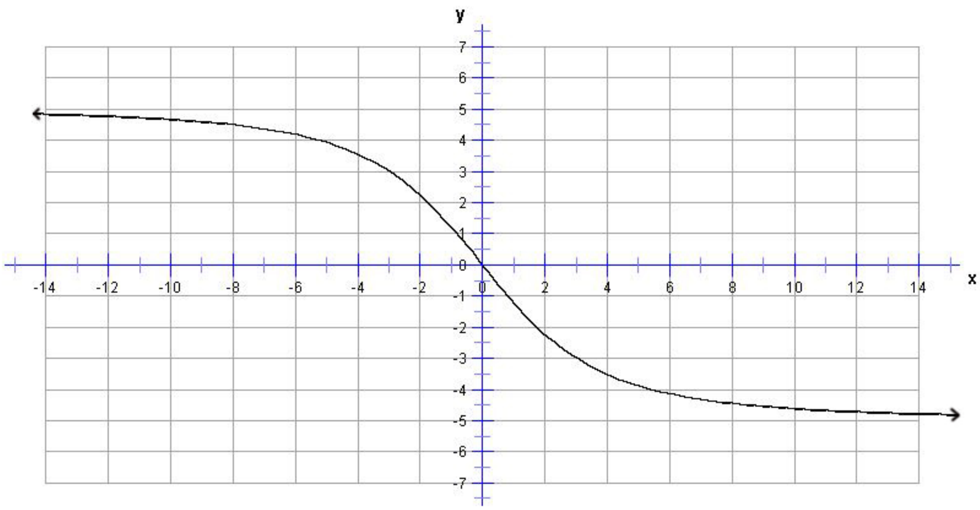
All Real Numbers because, for the entire domain, the bottom is always greater than or equal to 4, while the top can be any real number.

2) Determine the intercepts and asymptotes – The y and x intercepts would both be $(0,0)$. Since there are no restrictions, there are no vertical asymptotes. However, there are "sneaky" horizontal asymptotes that exist. Remember that horizontal asymptotes exist with the degree of the top equals the degree of the bottom and, in that case, the asymptote would be at the ratio of the leading coefficients. This, however, does not appear to apply to this problem, but it actually does apply from a limit perspective. Consider x going to either positive or negative infinity. In both cases, the + 16 under the radical becomes irrelevant so the bottom basically becomes the absolute of x. Therefore, when x approaches negative infinity, y approaches +5 and, when x approaches positive infinity, y approaches –5. This means that there are two horizontal asymptotes, $y=5$ and $y=-5$.

3) Find all values of $f'(x)$ and $f''(x)$ that equal 0 or are undefined to locate, relative extrema, inflection points – Find the first derivative using either the quotient or product rule. Using the product rule, $f'(x)=\frac{5x^2}{\sqrt{(x^2+16)^3}}-\frac{5}{\sqrt{x^2+16}}\to\frac{-16}{\sqrt{(x^2+16)^3}}$. Setting $f'(x)$ equal to zero and clearing fractions yields $-16=0$ which indicates that there are no solutions. Therefore there are no minima or maxima for this function. Setting $f''(x)=\frac{24x}{\sqrt{(x^2+16)^5}}$ equal to 0 and clearing fractions yields the equation $24x=0$. Therefore, there is an inflection at $x=0$ which give you an inflection point at $(0,0)$.

4) Use the second derivative on all intervals to determine concavity – For concavity, the intervals are determined by vertical asymptotes and inflection points. Since there are no vertical asymptotes, only the one inflection point break up the graph into these two intervals… $x<0$ and $x>0$. Keep in mind that you can choose ANY point in an interval to test the entire interval. Therefore, in most cases, you can choose integers to test the sections which will make the math easier. Choosing $x=-1$ and substituting it into the second derivative yields $f''(-1)=\frac{24(-1)}{\sqrt{((-1)^2+16)^5}}=\frac{-24}{289\sqrt{17}}$, which is negative. This means that the graph is concave down in the left interval. Choosing $x=1$ and substituting it into the

second derivative yields $f''(1)=\dfrac{24(1)}{\sqrt{\left((1)^2+16\right)^5}}=\dfrac{24}{289\sqrt{17}}$, which is positive. This means that the graph is concave up in the right interval. You now have enough information to make a very accurate sketch of this graph:




6) The carpet manufacturer, Carpets R Us, using the following formulas to calculate their profit on a custom-made carpet: Cost per color/shape = price per square yard times the area of that color/shape, Profit = Cost times the profit factor. For custom made carpets, the profit factor ranges from –10 to +10, where –10 represents the least profitable and +10 represents the most profitable custom carpet based on color, type, shape, and installation. A customer has requested a custom carpet designed to consist of a red, triangular region in the center with yellow, semicircular regions attached to each side of the triangle. Each leg of the triangle is twice the length of the base, x. If the cost of the red carpet 8 divided by x dollars per square yard and the cost of the yellow carpet is x divided by 4 dollars per square yard, and the profit factor for manufacturing the semicircular yellow carpets is –2 while the profit factor for the triangular red carpet is +6, find the exact, and approximate (rounded to one decimal place) length of the base of the red triangle that maximizes the total profit for the entire carpet.

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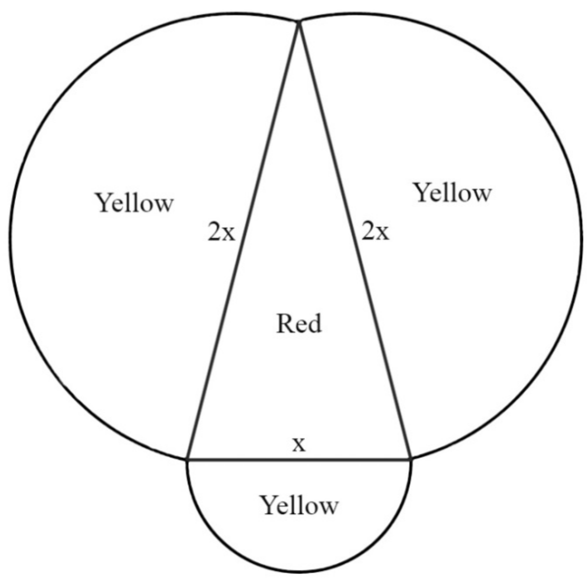
Answer: Follow these 4 steps:

1) Make a sketch and label all important information in the sketch:




2) Write a general equation that describes the problem and determine all domains:

Since we are describing the physical dimensions of real objects, all domains are any real number greater than zero. The area of the yellow carpet is comprised of three semicircles. Two of the semicircles are the same size, radius r, so the combined area would be a complete circle with its area given by $A_1 = \pi r^2$. The third semicircle has a diameter of d, so its area, $A_2$, would be half that of the full circle or $A_2 = \frac{1\pi}{2}\left(\frac{d}{2}\right)^2$.

Therefore, the total area of the yellow carpet would be $A_y = A_1 + A_2$ or

$A_y = \pi r^2 + \frac{\pi}{2}\left(\frac{d}{2}\right)^2 \rightarrow A_y = \frac{8\pi r^2 + \pi d^2}{8}$. Finding the area of the red triangle requires the formula $A = \frac{1}{2} \cdot b \cdot h$. Therefore, the red carpet area is $A_R = \frac{1}{2} \cdot b \cdot h$. Finally, the total area, $A$, of both the red and yellow carpet would be $A = A_y + A_R$ which becomes

$$A = \frac{8\pi r^2 + \pi d^2}{8} + \frac{1}{2} \cdot b \cdot h \rightarrow A = \frac{8\pi r^2 + \pi d^2 + 4bh}{8} \rightarrow A = \pi r^2 + \frac{\pi d^2}{8} + \frac{bh}{4}.$$

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3) Try to manipulate that equation to reduce it to a single variable:

In this case, try to write $A = \pi r^2 + \frac{\pi d^2}{8} + \frac{bh}{4}$ in terms of only x, the base of the red triangle. The first step would be to find the height, h, of the triangle using the Pythagorean Theorem. Using the right triangle formed by this height yields $h^2 + \left(\frac{x}{2}\right)^2 = (2x)^2 \rightarrow h^2 + \frac{x^2}{4} = 4x^2 \rightarrow h^2 = \frac{15x^2}{4} \rightarrow h = \frac{x\sqrt{15}}{2}$. Knowing, from the information given, that $d = x$, $b = x$, and $r = x$ and then substituting these values into the area formula yields $A = \pi x^2 + \frac{\pi x^2}{8} + \frac{x^2\sqrt{15}}{4}$. Simplifying yields $A = \frac{9\pi x^2}{8} + \frac{x^2\sqrt{15}}{4}$. The cost of carpet depends on the area, type, and color and is determined by the formula Cost = Price per square yard times the Area. So, if the price per square yard of the yellow carpet is $\frac{x}{4}$ dollars and the price per square yard of the red carpet is $\frac{8}{x}$ dollars, then the total cost would be $C = price(area) \rightarrow C = \frac{x}{4} \cdot \frac{9\pi x^2}{8} + \frac{8}{x} \cdot \frac{x^2\sqrt{15}}{4} \rightarrow C = \frac{9x^3\pi}{32} + 2x\sqrt{15}$. The profit realized by the carpet company, including manufacturing and installation, is determined by a profit factor, F. Since the profit factor depends on the amounts, types, colors, and shapes of the carpet, it is measured on a scale of –10 to +10. In this case, the very labor intensive, semicircular, yellow carpet results in a profit factor of –2, while the less labor intensive, triangular, red carpet results in a profit factor of +6. The profit realized by the company is modeled by the formula Profit = Cost times Profit Factor. Use the formula $P = C(F)$ which gives you

$$P = -2 \cdot \frac{-9x^3\pi}{16} + 6 \cdot x\sqrt{15} \rightarrow P = \frac{-9x^3\pi}{16} + 12x\sqrt{15}.$$

4) Use derivatives to find maximums or minimums:

Since we are trying to maximize the total company profit of the carpet, P, as it relates to the length of the base of the red triangle, x, we should take the derivative of P with respect to x. The derivative yields the equation $\frac{dP}{dx} = \frac{-27x^2\pi}{16} + 12\sqrt{15}$. The total profit for the carpet, P, would be maximized when $\frac{dP}{dx}=0$. Therefore, solve $\frac{-27x^2\pi}{16}+12\sqrt{15}=0$. Clearing fractions yields $-27\pi x^2+192\sqrt{15}=0$. Solving for x yields $x^2=\frac{192\sqrt{15}}{27\pi} \to x=\pm\frac{8\sqrt[4]{15}\cdot\sqrt{\pi}}{3\pi}$. Therefore, either $x=-\frac{8\sqrt[4]{15}\cdot\sqrt{\pi}}{3\pi}$ or $x=\frac{8\sqrt[4]{15}\cdot\sqrt{\pi}}{3\pi}$. The length of the base of the red carpet, x, cannot be negative.

Therefore, the length of the carpet that maximizes the profit would be $x=\frac{8\sqrt[4]{15}\cdot\sqrt{\pi}}{\pi}$. Rounded to one decimal place, gives you a final answer of approximately 3.0 yards.