Teaching Notes for Calculus
Homework #14
Definite and Indefinite Integrals

HAND OUT INTEGRAL RULES – Tell the class that, just like multiplication and division are the mathematical opposites of each other, so are derivatives and integrals. This means that an integral is really an "anti-derivative." Discuss the difference between indefinite and definite integrals!

Definite Integrals represent the area between the curve and the x axis. If the graph is above the axis, the area is positive, below the x axis the area is negative!

Indefinite Integrals of a function determine what the original function might have look like so that, if you took its derivative, you would get the function you started with. The answer to an indefinite integral is never exact unless addition information is provided. Therefore, you must always add an unknown constant, c, to the end of your answer. Indefinite integrals are basically the opposite of derivatives. For example, if the derivative of the distance is velocity and the derivative of velocity is acceleration, then the integral of acceleration is velocity and the integral of velocity is distance. Everything works in reverse.

The Power Rule for derivatives works for integrals but in reverse (You increase the power by one and divide by that new power. Also, the u, du substitutions that you did for composite functions for derivatives also work backwards, and can be far more difficult to determine. You can't use one of the standard integral rules unless you also have the derivative of the function inside, in the problem. If you don't have it, you must try your best to create it. Once you have it, it will disappear once you take the integral, just like derivative of the inside function just shows up when taking a composite function derivative.

Make sure you discuss the basic rules of integrals…like, if you switch the position of the numbers for the bounds, it simply changes the sign of the original answer. Another basic rule is that if you need to find a definite integral from 2 to 9, you can take the answer from say 2 to 5 and just add it to the answer from 5 to 9. Another basic rule is that constants can be left inside or brought outside of an integral without changing the final answer. Finally, the answer to a definite integral, where the lower and upper bounds are the same, is always 0.

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Classroom Examples

1) Find $\int\left(-4x^{\frac{2}{5}}+5x^2-7\right)dx$ and check the result by differentiation.

Answers: Recognize that this is an indefinite integral and then use the Power Rule in reverse to get $f(x)=\dfrac{-20x^{\frac{7}{5}}}{7}+\dfrac{5x^3}{3}-7x+c$. Differentiating this answer yields, $f'(x)=-4x^{\frac{2}{5}}+5x^2-7$, thus proving that your integration was correct!

2) Find the particular solution that satisfies $f''(x)=-36x^{\frac{-5}{2}}, f'(4)=-4, f(9)=1$

Answer: In order to find the original function, we will have to take the integral twice. First, $\int f''(x)=f'(x)=24x^{\frac{-3}{2}}+c$. However, in this problem, we are provided with additional information so that we can find *c*. Since $f'(4)=-4$, substitution yields $-4=\dfrac{24}{\sqrt{4^3}}+c$, therefore, $c=-7$ so $f'(x)=24x^{\frac{-3}{2}}-7$. Now take another integral. So, $\int f'(x)=f(x)=-48x^{\frac{-1}{2}}-7x+k$. However, once again, we are provided with addition information so that we can find *k*. Since $f(9)=1$, substitution yields $1=\dfrac{-48}{\sqrt{9}}-7(9)+k$, therefore, $k=80$. Therefore, the original function was $f(x)=-48x^{\frac{-1}{2}}-7x+80$.

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3) Evaluate $\int_{-2}^{3} x^5 dx$.

Answer: Recognize that this is a definite integral. Therefore there won't be a constant and the integral will have to be evaluated at the bounds of integration. $\int_{-2}^{3} x^5 dx = \left.\frac{x^6}{6}\right|_{-2}^{3}$.

Evaluating this expression at the upper bound of 3 gives you $\frac{243}{2}$ and then evaluating the expression at the lower bounds of –2 gives you $\frac{32}{3}$. Finally, you subtract the answer for the lower bound FROM the upper bound answer. Therefore, the final answer is $\frac{243}{2} - \frac{32}{3} = \frac{665}{6}$.

4) Find $\int \frac{3x^2 - 5x + 7}{\sqrt[3]{x^2}} dx$ and check the result by differentiation.

Answers: Recognize that this is an indefinite integral, separate the integral into three terms, simplify each term, and then use the Power Rule in reverse to get the following:

$$\int \left(\frac{3x^2}{x^{\frac{2}{3}}} - \frac{5x}{x^{\frac{2}{3}}} + \frac{7}{x^{\frac{2}{3}}} \right) dx \rightarrow \int \left(3x^{\frac{4}{3}} - 5x^{\frac{1}{3}} + 7x^{\frac{-2}{3}} \right) dx \rightarrow f(x) = \frac{9x^{\frac{7}{3}}}{7} - \frac{15x^{\frac{4}{3}}}{4} + 21x^{\frac{1}{3}} + c.$$

Differentiating this answer yields, $f'(x) = 3x^{\frac{4}{3}} - 5x^{\frac{1}{3}} + 7x^{\frac{-2}{3}}$, basic algebraic manipulation of this equation results in $f'(x) = \frac{3x^2 - 5x + 7}{\sqrt[3]{x^2}}$, thus proving that your integration was correct!

5) The rate of growth $\frac{dP}{dt}$ of a population of viruses varies directly as the cube root of the time, t, squared, where P is the population size and t is the time in days for all t such that $0 \le t \le 6$. The initial size of the population is 300. After 1 day, the population

has grown to 600. What would the population be after 5 days, rounded to a whole number?

Answer: Remembering how to write variation equations from algebra, write an equation for the rate of growth, $\frac{dP}{dt}$. According to the information given, $\frac{dP}{dt} = k\sqrt[3]{t^2}$. Multiplying each side by dt yields $dP = k\sqrt[3]{t^2}\,dt$. Taking an indefinite integral of both sides of this equation yields $\int dP = \int k\sqrt[3]{t^2}\,dt \to P(t) = \frac{3kt^{\frac{5}{3}}}{5} + c$. We can find c because they told us that, at $t = 0$, $P = 300$. Substitution reveals that $c = 300$. We can also find k since we also know that, when $t = 1$, $P = 600$. Substitution reveals that $k = 500$. Therefore, the function that dictates population growth as a function of time is $P(t) = 300t^{\frac{5}{3}} + 300$. Evaluating this function when $t = 5$ yields 4686.0266 which rounds to 4686 viruses.

6) Evaluate $\int_1^2 (2x^2 - 3x + 5)dx$.

Answer: Recognize that this is a definite integral. Therefore there won't be a constant and the integral will have to be evaluated at the bounds of integration.

$\int_1^2 (2x^2 - 3x + 5)dx = \Big|_1^2 \left(\frac{2x^3}{3} - \frac{3x^2}{2} + 5x\right)$. Evaluating this expression at the upper bound of 2 gives you $\frac{28}{3}$ and then evaluating the expression at the lower bounds of 1 gives you $\frac{25}{6}$. Finally, you subtract the answer for the lower bound FROM the upper bound answer. Therefore, the final answer is $\frac{28}{3} - \frac{25}{6} = \frac{31}{6}$.

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7) Find $\int (5m^2 + \sec m \tan m) dm$ and check the result by differentiation.

Answers: Recognize that this is an indefinite integral, separate the integral into two terms, use the Power Rule in reverse on the first term and look for a derivative trig rule to use in reverse for the other term: $\int 5m^2 dm + \int (\sec m \tan m) dm \rightarrow \frac{5m^3}{3} + \sec m + c$, therefore $f(m) = \frac{5m^3}{3} + \sec m + c$. The $\sec m$ part of this answer comes from the fact that the derivative of $\sec m$ is $\sec m \tan m$. Differentiating this answer yields $f'(m) = 5m^2 + \sec m \tan m$, thus proving that your integration was correct!

8) On Mercury, the acceleration due to gravity is – 3.7 meters per second per second. A stone is dropped from a cliff on Mercury and hits the surface of the planet 30 seconds later. How far did it fall? What was its velocity at impact?

Answers: As stated earlier, since the derivative of distance is velocity and the derivative of velocity is acceleration, then the reverse of this must be true for integration. The integral of acceleration is velocity and the integral of velocity is distance. Therefore, calling the acceleration *a*, and the velocity *v*, the integral of *a*, with respect to time, t, is the velocity, *v* or $v(t) = \int (a) dt$. However, on the planet Mercury, the acceleration due to gravity is constant and does not vary with the time. Therefore, treating *a* as a constant, yields $v(t) = \int (a) dt = a \cdot t + c$ Therefore, calling the distance *s*, and keeping the velocity as *v*, the integral of *v* is *s*, or $s(t) = \int (v) dt = \int (a \cdot t + c) dt = \frac{at^2}{2} + c \cdot t + k$. The students should recognize this formula from previous math classes as it is the formula for distance in projectile motion problems. In the formula $s(t) = \frac{at^2}{2} + c \cdot t + k$, *a* is the acceleration, *c* is the initial velocity, and *k* is the initial height. Therefore, to answer the first question, use $a = -3.7$, $c = 0$, $k = 0$, and $t = 30$. The initial velocity is zero because the stone was just dropped and we can make the height of the cliff zero, just as a reference point. Substituting all of these values into the distance formula gives you –

1665 for the distance the stone fell. To answer the second question, knowing that the stone hit the ground after 30 seconds, we can calculate the stone's final velocity at impact by simply substituting $a = -3.7$ and $t = 30$ into the velocity equation. This yields -111 meters per second. The answer is negative because the stone is traveling towards the surface of the planet and not away from it.