Teaching Notes for Calculus
Homework #15
Area Limit & Riemann Sums & the Midpoint, Trapezoidal, & Simpson's Rules

The **Area Limit-Sum** method calculates the area under a curve: $Area = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$

where $\Delta x = \frac{b-a}{n}$ and $c_i = a + \frac{(b-a)i}{n} = a + i\Delta x$ where b and a are the endpoints on the x axis.

A subtle restatement of this unique method for finding the area between the x axis and a given curve can be used to evaluate a definite integral. This restatement, also known as a **Riemann Sum**, equals the corresponding definite integral, and can be written as $\int_a^b f(x)dx = \lim_{\Delta x\to 0}\sum_{i=1}^{n} f(c_i)\Delta x$, where $\Delta x = \frac{b-a}{n}$ and $c_i = a + \frac{(b-a)i}{n} = a + i\Delta x$. The Δx is the width of each partition and differs from the Area Limit-Sum formula by having this width approach zero. If the width is approaching 0, then number of partitions, *n,* is approaching infinity. Just as the derivate was defined as a limit that approached 0, the Riemann Sum restatement results in a limit approaching 0 and is considered to be the definition of a definite derivative. While this may appear to just be a subtle, restatement, it is actually very important as the Area Limit-Sum formula only applies to areas under a curve while Riemann Sum definite integrals can be applied to many applications including work, arc lengths, areas, volumes, surface areas, average values, etc. *Basically, any entity which can be determined by the product of two variable*

Both of these formulas will work for finding the area between the y axis any function in terms of y, f(y), if you just swap everything around, the axes and the variables, properly. This can be VERY tricky in terms of keeping everything organized so take the appropriate amount of time explaining this concept in the classroom examples.

Review the four common summation formulas from geometry and pre-calculus as they are needed in order to solve many of the problems for this lesson

1. $\sum_{i=1}^{n} c = cn$, c is a constant

2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

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There are 3 commonly used methods for approximating the area for definite integrals:

1) **The Midpoint Rule**: You basically takes each interval and turns it into a rectangle where the height of each rectangle is the function value at the middle of each interval. You will be given the number of intervals, n. Make $\Delta x = \frac{b-a}{n}$ and $c_i = a + \left(i - \frac{1}{2}\right)\Delta x$ or the midpoint of each interval, and then calculate $Area \approx \sum_{i=1}^{n} f(c_i)\Delta x$

2) **The Trapezoidal Rule**: You basically takes each interval and turns it into a trapezoid where the left and right bases of the trapezoid are the function values taken at the beginning and end of each interval. Again, you will be given the number of intervals, n.

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

3) **Simpson's Rule**: This is the most complicated method as, instead of using a straight line, as both the midpoint and trapezoidal methods do to approximate the top of each interval, you can use a quadratic curve. In many cases, this will give you a better answer than the trapezoidal method but it could give you a worse answer. Also, the number of intervals, n, MUST be an even number!

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)].$$

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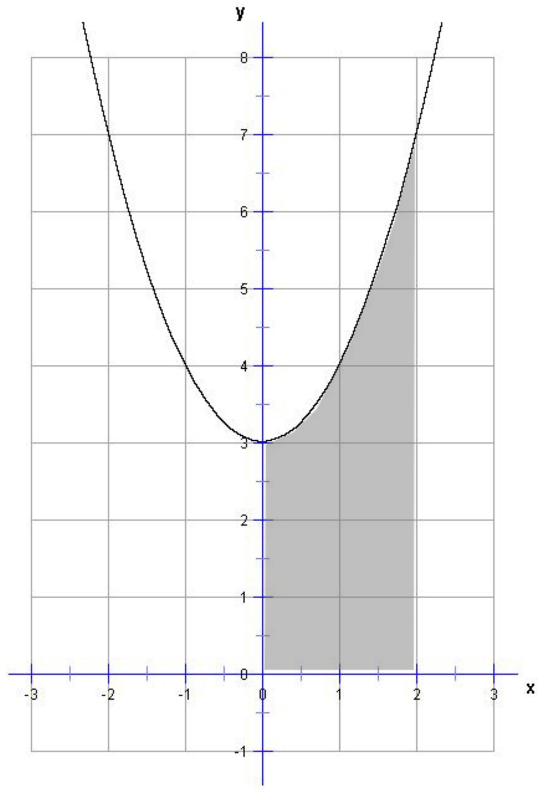
Classroom Examples



1) Use the Area Limit-Sum method to find the area bounded by the function $y = x^2 + 3$ and the x axis over the interval [0,2] and sketch the region.

$Area = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$. Breaking the interval [0,2] into n

subintervals each of width $\Delta x = \frac{2}{n}$ and choosing

$c_i = \frac{2i}{n}$ for each

interval gives

$$Area = \lim_{n\to\infty}\sum_{i=1}^{n}((\frac{2i}{n})^2 + 3)(\frac{2}{n}) = \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left(\frac{4i^2}{n^2} + 3\right) =$$

$$\lim_{n\to\infty}\frac{2}{n}\left(\sum_{i=1}^{n}\frac{4i^2}{n^2} + \sum_{i=1}^{n}3\right) = \lim_{n\to\infty}\frac{2}{n}\left(\frac{4}{n^2}\sum_{i=1}^{n}i^2 + \sum_{i=1}^{n}3\right) =$$

$$\lim_{n\to\infty}\frac{2}{n}\left(\frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} + 3n\right) = \lim_{n\to\infty}\left(\frac{4(n+1)(2n+1)}{3n^2} + 6\right) =$$

$$\lim_{n\to\infty}\left(\frac{26}{3} + \frac{4}{n} + \frac{4}{3n^2}\right) = \frac{26}{3}$$

2) Use the Midpoint Rule with n = 3 to approximate the area of the region bounded by the function $f(x) = 2x^3 - 1$ over the interval [1, 4].

$$Area \approx \sum_{i=1}^{n} f(c_i)\Delta x \approx \sum_{i=1}^{3}\left(2(c_i)^3 - 1\right)\cdot 1 = 2\left(\frac{3}{2}\right)^3 - 1 + 2\left(\frac{5}{2}\right)^3 - 1 + 2\left(\frac{7}{2}\right)^3 - 1 = \frac{483}{4}$$

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3) Evaluate $\int_3^8 4x\,dx$ using the Riemann Sum definition method.

$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(c_i)\Delta x$. Breaking the interval [3,8] into n subintervals each of width $\Delta x = \frac{5}{n}$ and choosing $c_i = 3 + \frac{5i}{n}$ for each interval gives

$$\int_3^8 4x\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} 4\left(3+\frac{5i}{n}\right)\left(\frac{5}{n}\right) = \lim_{n\to\infty} \frac{60}{n}\sum_{i=1}^{n}\left(1+\frac{100i}{n^2}\right) = \lim_{n\to\infty}\frac{60}{n}\left(\sum_{i=1}^{n}1+\frac{100}{n^2}\sum_{i=1}^{n} i\right) =$$

$$\lim_{n\to\infty}\left(60+50+\frac{50}{n}\right)=110$$

4) Approximate $\int_2^4 \sqrt{x^4 - 10}\,dx$ to three decimal places using both the Trapezoidal Rule and Simpson's Rule with $n = 4$.

Trapezoid: 19.835

$$\int_2^4 \sqrt{x^4+9}\,dx = \frac{4-2}{8}\left(\sqrt{(2)^4+9}+2\sqrt{(2.5)^4+9}+2\sqrt{(3)^4+9}+2\sqrt{(3.5)^4+9}+\sqrt{(4)^4+9}\right)$$

Simpson's: 19.739

$$\int_2^4 \sqrt{x^4+9}\,dx = \frac{4-2}{12}\left(\sqrt{(2)^4+9}+4\sqrt{(2.5)^4+9}+2\sqrt{(3)^4+9}+4\sqrt{(3.5)^4+9}+\sqrt{(4)^4+9}\right)$$

5) Use the Area Limit-Sum method to find the area bounded by the function $f(y) = 2y^2 + 1$ and the y axis over the interval $0 \le y \le 2$ and sketch the region.

$Area = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta y$. Breaking the interval $0 \le y \le 2$ into n subintervals each of width $\Delta y = \frac{2}{n}$ and choosing $c_i = \frac{2i}{n}$ for each interval gives:

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$$Area = \lim_{n\to\infty}\sum_{i=1}^{n}\left(2\left(\frac{2i}{n}\right)^2+1\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty}\left(\frac{16}{n^3}\sum_{i=1}^{n} i^2 + \frac{2}{n}\sum_{i=1}^{n} 1\right) = \lim_{n\to\infty}\left(\frac{16}{n^3}\frac{n(n+1)(2n+1)}{6}+2\right) = \lim_{n\to\infty}\left(\frac{22}{3}+\frac{8}{n}+\frac{8}{3n^2}\right) = \frac{22}{3}$$