# Teaching Notes for Calculus
# Homework #16
# The Fundamental Theorems of Calculus, Integration by Substitution and Change of Variable, and Displacement Versus Total Distance

**THEOREM 4.9 The Fundamental Theorem of Calculus**

If a function $f$ is continuous on the closed interval $[a, b]$ and $F$ is an antiderivative of $f$ on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

You should explain to the students WHY you don't need the constant, C, when evaluating a definite derivative. The C's cancel out from the integral at b minus the integral at a!

Discuss definite integrals on absolute value functions. If the "bounce" occurs within the interval from a to b, then you must create two different integrals using different equations and bounds. You need one for the left side of the bounce, and the other for the right because the function isn't differentiable at the point of the sharp bounce. Keep in mind that this rule applies to any point where the left and right derivative limits are not equal.

Emphasize again that definite integrals can be used to evaluate virtually any multiplication application, even when the products vary like for $D = R \cdot T$, $W = F \cdot D$, $Mean = \frac{1}{n} \cdot Sum$ etc.

**Important** Explain the difference between total distance traveled and displacement!

**Displacement** is the final location of the object and is determined by the sum of all of the areas (+ above the x-axis and – below the x-axis) as determined by the total integral!

**Total Distance** is how far an object travels and must be determined by forcing all areas (above and below the x axis) to be positive!

**THEOREM 4.11 The Second Fundamental Theorem of Calculus**

If $f$ is continuous on an open interval $I$ containing $a$, then, for every $x$ in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

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**The Second Fundamental Theorem of Calculus** allows the bounds of a definite integral to be functions themselves based on a different variable (change in variable):

$\int_a^x f(t)dt$ = the integral of the function with respect to t which is then evaluated at the function x (like doing composite functions), minus the integral evaluated at the constant *a*.

**Integration by Substitution** allows more complicated integration to be performed:

$$\int f(u)du = F(u) + C \text{ where } u = g(x) \text{ and } du = g'(x)$$

As mentioned before, you always need to create the derivative of function inside the function. To help accomplish this it is important to know that you can place any number you need inside the integral as long as you place its reciprocal in front of the integral.

**Integration by Change of Variable** works just like substitution with the added benefit being that you have the option of solving your u substitution for x to integrate more complex problems. Don't even attempt to explain what that means at this point in the lecture, wait until you actually walk the students through a classroom example that requires this concept.

## Classroom Examples

1) Evaluate $\int_1^6 \left|16 - x^2\right| dx$

Answer: Begin by graphing the function and noticing that this absolute value function "bounces" at x = 4. You now must create two separate functions, one that represents the left side of the break and the other that represents the right side. According to the rules of absolute value functions, the function on the left would be $16 - x^2$ while the one on the right would be

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$x^2 - 16$. This means that $\int_1^6 \left|16 - x^2\right| dx = \int_1^4 \left(16 - x^2\right) dx + \int_4^6 \left(x^2 - 16\right) dx$. Integration yields

$\Big|_1^4 \left(16x - \frac{x^3}{3}\right) + \Big|_4^6 \left(\frac{x^3}{3} - 16x\right)$ which evaluates as $\left(\frac{128}{3} - \frac{47}{3}\right) + \left(\frac{-72}{3} + \frac{128}{3}\right) = \frac{137}{3}$

2) Find $f(x) = \int_{\pi/6}^{x} \csc t \cot t\, dt$ and then demonstrate the Second Fundamental Theorem of Calculus by differentiating the result.

Answer: According to the equation, $f(x)$, the original function was $f(t) = \csc t \cot t$ and $a = \frac{\pi}{6}$.

Using the trig rules for integrals, $f(x) = \Big|_{\pi/6}^{x} -\csc(t)$. This means that $f(x) = -\csc(x) + 2$.

Differentiating $f(x)$ yields $\frac{d}{dx}[f(x)] = \frac{d}{dx}[-\csc x + 2] = \frac{d}{dx}\left[\Big|_{\pi/6}^{x} -\csc(t)\right] =$

$\frac{d}{dx}\left[\int_{\pi/6}^{x} \csc t \cot t\, dt\right] = \frac{d}{dx}\left[\int_{\pi/6}^{x} f(t)dt\right] = \frac{d}{dx}\left[\int_{a}^{x} f(t)dt\right] = f(x)$

3) At 10am, a dam begins to fail and starts leaking. The rate of the flow of water from the failing dam is $(38 + 5t)$ gallons per minutes where t is the time in minutes. How much water flows out from 10am to 11:30am? How much water flows out from 11:30am to 1pm? What do you notice when you compare these two answers?

Answers: Just as the distance traveled equals the rate of motion times the time traveled, the amount of water flow equals the rate of water flow times the amount of time it flowed. This means that we can find a solution to this problem through integration. Create the equation $W(t) = \int_{t_0}^{t_f} R(t) \cdot dt$. Substitution for the first part of the question yields $W(t) = \int_0^{90} (38 + 4t)dt$.

Integration yields $W(t) = \Big|_0^{90} \left(38t + 2t^2\right)$. Evaluating this definite integral yields

$\Big|_0^{90} \left(38t + 2t^2\right) = (19620 - 0)$ or 19,620 gallons of water flow between 10am and 11:30am. Now repeat this same process for the time period 11:30am and 1pm. Therefore,

$W(t) = \int_{90}^{180} (38 + 4t)dt$ and integration yields $\Big|_{90}^{180} \left(38t + 2t^2\right) = (71640 - 19620)$ or 52,020 gallons of water flow between 11:30am and 1pm. What this data tells you is that the size of the break in the dam is increasing over time since the amount of water flowing out of the dam nearly tripled from the first 90 minutes to the second 90 minutes.

4) Evaluate $\int -7x^2(x^3-5)^4\,dx$

Answer: There are technically two possible ways to approach this problem. The first method, which you DO NOT want to attempt, would be to write out all of the terms and then expand all of the multiplication. Depending on the size of the problem, this will be tedious and time consuming at best and next to impossible at worst. The method you want to try first would be a reverse u du substitution, the opposite of what you learned to do for derivatives. For this to work, you need the problem to be multiplication and then, usually, choose what's inside any parentheses to be u. In this case, $u = x^3 - 5$. Therefore, $du = 3x^2 dx$. Remember, however, that we are NOT differentiating. With integration, you MUST be able to replace every part of the problem that contains x, with the u and the du. HINT: you know you are headed in the right direction if the x part of the du matches the x's outside the parentheses in the original problem. Since $du$ and $-7x^2 dx$, which is on the outside of the parentheses, both have an $x^2$ in them, this method will work! The only issue is that one contains a 3 while the other has a –7, but this can be fixed. You can always take a number that is inside an integral and place it in front of it, just as you could with derivatives. That takes care of the –7. In order to put a number inside an integral, all you have to do is multiply what's in front of the integral by that number's reciprocal. For this problem, you will pull out the –7 and then replace it with a 3 as long as you multiply the – 7 on the outside by $\frac{1}{3}$. The resulting integral would be $\frac{-7}{3}\int 3x^2(x^3-5)^4\,dx$ and you are now ready to perform the u du substitution which gives you $\frac{-7}{3}\int u^4\,du$. Use the power rule to get $\frac{-7u^5}{15}+c$. Now put x back into the problem be remembering that, in the beginning, you stated that $u = x^3 - 5$. This gives you a final answer of $\frac{-7\left(x^3-5\right)^5}{15}+c$.

5) Evaluate $\int \frac{4x^5}{\sqrt{x^6+3}}\,dx$

Answer: Again, this integral looks impossible but, upon closer inspection, if you made $u = x^6 + 3$, which makes $du = 6x^5 dx$, then a u du substitution will work! In this case, we need the 4 to be turned into a 6. Since 4 and 6 both have 2 as a factor, just take out the 2 from the 4 and leave the other 2 inside the integral. Now all you need is to multiply the inside by 3 and the outside by one-third. Following these steps leads to this new, manipulated integral: $\frac{2}{3}\int \frac{4x^5}{\sqrt{x^6+3}}dx$. You are now ready to perform the u du substitution which gives you $\frac{2}{3}\int u^{\frac{-1}{2}}du$. Use the power rule to get $\frac{4\sqrt{u}}{3}+c$. Now put x back into the problem be remembering that, in the beginning, you stated that $u=x^6+3$. This gives you a final answer of $\frac{4\sqrt{x^6+3}}{3}+c$.

6) Solve the differential equation $\frac{dy}{dx}=\frac{3x^2-10x}{(x^3-5x^2+4)^2}$

Answer: To solve a differential equation, you must multiply both sides by dx as you cannot perform integration on x with there being a dx attached to it through multiplication. Multiplying by dx yields $dy=\frac{3x^2-10x}{(x^3-5x^2+4)^2}dx$. Solving a differential equation means finding y. Therefore, you must integrate both sides which gives you $\int dy=\int \frac{3x^2-10x}{(x^3-5x^2+4)^2}dx$. The left side of the equation will just become y + c, but the right side looks like it's another candidate for a u du substitution if you made $u=x^3-5x^2+4$, which makes $du=\left(3x^2-10x\right)dx$. In this case, the u du substitution works perfectly as no manipulations are necessary. Your new equation will resemble $y+c_1=\int u^{-2}du$, which, after integrating the right side, becomes $y+c_1=\frac{-1}{u}+c_2$. Putting the x's back in and combining the two constants into one, called c, gives you the final answer of $y=\frac{-1}{x^3-5x^2+4}+c$

7) Evaluate $\int \frac{3\sec^2 x}{2\tan^4 x}dx$

Answer: In the last example, the u du substitution was not only obvious, but worked perfectly. Many integrals that require a u du substitution are NOT obvious and require a lot of thought and/or manipulations to make them work. In this example, you need to recall that the derivative of $\tan x$ is $\sec^2 x$. Knowing this, if you made $u = \tan x$, it would make $du = \sec^2 x dx$. This change in variable results in $\frac{3}{2}\int u^{-4} du$ and integrating this result yields $\frac{-1}{2}u^{-3} + c$. Remembering that you made $u = \tan x$ gives you the final result of $\frac{-1}{2\tan^3 x} + c$.

8) Evaluate $\int \frac{2x^2 - 4x + 2}{\sqrt{2x-2}} dx$

Answer: These last two problems are examples of just how tricky and complicated these problems can be. Upon first inspection, it appears that performing a u du substitution with $u = 2x^2 - 4x + 2$ would work because $du = (4x-4)dx$ which appears like it would fit somehow under the radical. However, the dx is not under the radical so this substitution will not work. Hint: when deciding how and what to use for the u du substitution, the issues you might want to address first are making sure that you have eliminated an radical issues and also making sure that your du will be able to replace your dx. Keeping that in mind, it might be a good idea to make $u = 2x - 2$ since that will take care of the radical. This might work, it might not. Sometimes you just have to keep trying until you get it just right. Continuing, this would make $du = 2dx$ which will take care of du going in for dx. However, you still have the $2x^2 - 4x + 2$ to deal with. Since the ultimate goal, at this point is to remove all of the x's and replace them with u's, we could try squaring u. Since $u = 2x - 2$, $u^2 = 4x^2 - 8x + 4$. With a few modifications, this is actually going to work! First, we need to multiply the inside by 2 in order for du to fit perfectly. Second, we will have to multiply the $2x^2 - 4x + 2$ by 2 in order for the $u^2$ to fit perfectly. Therefore, we can multiply the inside of the integral by 4 as long as we multiply the outside by one-fourth. This will modify the problem so it resembles $\frac{1}{4}\int \frac{2(4x^2 - 8x + 4)}{\sqrt{2x-2}} dx$. You are now ready to perform the u du substitution which gives you $\frac{1}{4}\int u^2 u^{\frac{-1}{2}} du = \frac{1}{4}\int u^{\frac{3}{2}} du$. Use the power rule to get $\frac{\sqrt{u^5}}{10} + c$.

Now put x back into the problem be remembering that, in the beginning, you stated that $u = 2x - 2$. This gives you a final answer of $\frac{\sqrt{(2x-2)^5}}{3} + c$.

9) Evaluate $\int x\sqrt{2x-1}dx$

Answer: This last example is the most challenging. This is due to the fact that you won't be able to successfully perform the integration whether you make $u = x$ or $u = 2x - 1$. If you don't want to take my word for this, I encourage you to try either of those scenarios and discover the impossibilities for yourself. The trick in this problem, as I previously stated, is dealing creatively with the radical. In this case, you could try to eliminate the radical altogether by making $u^2 = 2x - 1$. Now we have to hope that du works out. If $u^2 = 2x - 1$, then $2udu = 2dx$ or $udu = dx$. Believe it or not, you can now remove all of the x's and replace them with u's by simply taking your $u^2 = 2x - 1$ and solving it for x to get $x = \frac{u^2+1}{2}$. Substitution yields $\int\left(\frac{u^2+1}{2}\right)u \cdot udu$. While this might look like it can't be integrated, if you multiply and expand what's inside the integral and take a 2 out of the bottom to get $\frac{1}{2}\int\left(u^4+u^2\right)du$. This integral can be turned into two integrals to get $\frac{1}{2}\left[\int u^4du + \int u^2du\right]$. Perform the integration to get $\frac{u^5}{10} + \frac{u^3}{6} + c$ and put the x's back in to get the final answer of $\frac{(2x-1)^{\frac{5}{2}}}{10} + \frac{(2x-1)^{\frac{2}{3}}}{6} + c$. This, however, can be simplified:

$$\frac{3(2x-1)^{\frac{5}{2}} + 5(2x-1)^{\frac{2}{3}}}{30} + c \rightarrow \frac{(2x-1)^{\frac{2}{3}}\left(3(2x-1)+5\right)}{30} + c \rightarrow$$

$$\frac{(2x-1)^{\frac{2}{3}}(6x+2)}{30} + c \rightarrow \frac{(2x-1)^{\frac{2}{3}}(3x+1)}{15} + c.$$