Teaching Notes for Calculus
Homework #17
Mean Value Theorem for Integrals and the Average Value of a Function

THEOREM 4.10 Mean Value Theorem for Integrals

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b - a).$$

Explain how this is similar to the Mean Value Theorem they studied while doing derivatives. Remind the students that the original Mean Value Theorem guaranteed that if an expression is a function and continuous on an interval on x, then there must be at least one value, c, on that interval such that f(c) equals the mean value of the function over that interval.

The **Mean Value Theorem for Integrals** basically guarantees that if a function is continuous on an interval then there must be at least one value, c, on that interval such that the area of the rectangle formed by multiplying f(c) by the width of the interval equals the definite integral of the function over that interval.

T**he Average Value of a Function on an Interval** is basically a restatement of the Mean Value Theorem for Integrals…it is simply the Mean Value formula if you divide by the interval width, b – a. With regards to the area under a curve, the answer to $\frac{1}{b-a}\int_a^b f(x)dx$ is the average height of the function over the interval [a, b].

Definition of the Average Value of a Function on an Interval

If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Important Note This worksheet includes random definite integrals for extra practice, using any of the rules discussed thus far, including the addition of one new integral rule:

$\int \frac{du}{u} = \ln|u| + c$. Also mention that this is simply the reverse of the derivative rule for natural logs.

Classroom Examples

1) Find the value(s) of c guaranteed by the Mean Value Theorem for Integrals for $f(x) = x^2 + 4x - 3$ on the interval [2, 5].

Answer: The Mean Value Theorem states that $\int_a^b f(x)dx = f(c)(b-a)$, for all values of c on the interval [a, b]. Therefore, $\int_2^5 (x^2 + 4x - 3)dx = (c^2 + 4c - 3)(5-2)$. Integrating and simplifying yields $\left.\left(\frac{x^3}{3} + 2x^2 - 3x\right)\right|_2^5 = 3c^2 + 12c - 9$. Evaluating the integral yields $72 = 3c^2 + 12c - 9$, which simplifies to $3c^2 + 12c - 81 = 0$. Using the quadratic formula to solve yields the following answers: $c = -2 - \sqrt{31}$ or $c = -2 + \sqrt{31}$, but c can't be negative because it's not on the interval [2, 5]. Therefore, the answer is $c = -2 + \sqrt{31}$.

2) Evaluate $\int_3^7 \frac{-x^3\sqrt{3x^4 - 48}}{4} dx$

Answer: This integral will require a u du substitution. Making $u = 3x^4 - 48$ makes $du = 12x^3 dx$. Modifying the integral to $\frac{-1}{48}\int_3^7 12x^3\sqrt{3x^4 - 48}dx$ allows you to perfectly substitute u and du to get $\frac{-1}{48}\int_3^7 u^{\frac{1}{2}} du$. Integration and substituting the x's back in yields $\left.\frac{-1}{72}(3x^4 - 48)^{\frac{3}{2}}\right|_3^7$. Evaluating the integral gives you $\frac{-1}{72}\left(21465\sqrt{795} - 195\sqrt{195}\right) =$
$\frac{-7155\sqrt{795} + 65\sqrt{195}}{24}$

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3) Find the average value of $f(x)=6x-5x^2$ over the interval [–2, 4] and find all values of x in this interval for which the function equals the average value.

Answer: Average Value=$\frac{1}{b-a}\int_a^b f(x)dx$. Substitution yields: Average Value=$\frac{1}{4+2}\int_{-2}^{4}\left(6x-5x^2\right)dx$ which simplifies to: Average Value = $\frac{1}{6}\Big|_{-2}^{4}\left(3x^2-\frac{5x^3}{3}\right)=\frac{1}{6}\left(\frac{-176}{3}-\frac{76}{3}\right)=-14$. Setting the original function equal to the average value yields $-14=6x-5x^2$ and setting this equation equal to zero yields $5x^2-6x-14=0$. Solving yields $x=\frac{3\pm\sqrt{79}}{5}$. Both of these solutions are on the interval. Therefore, the function equals the average value when $x=\frac{3+\sqrt{79}}{5}$ or $x=\frac{3-\sqrt{79}}{5}$.

4) The relative intensity, I, of a tropical storm is jointly proportional to the $\cos t$ and square root of $1+\sin t$, where t is the time, in days, that the storm lasts. If the domain of I is $[0,2]$ and $I(0)=\frac{1}{2}$, find I as a function of t. If the amount of damage, D, caused by the storm over a given time period, in days, is directly proportional to the intensity, I, and $D\left(\frac{1}{2}\right)=2$, find D as a function of t. Find the damage caused by the storm during the first 6 hours and compare that answer to the damage caused between the 6th hour and 12th hour after the storm begins. Finally, find the average intensity of the storm over its two day life. Round all numerical answers to 3 decimal places.

Answers: $I=k\cos t\sqrt{1+\sin t}$. If $I(0)=\frac{1}{2}$, then $k=\frac{1}{2}$ so $I=\frac{\cos t\sqrt{1+\sin t}}{2}$. Also, if $D=k\cdot I$ and $D(\frac{1}{2})=2$, then $k=4$ so $D=4I$. Therefore, $D=2\cos t\sqrt{1+\sin t}$. In order to find the damage caused over a period of time, $\frac{dD}{dt}=2\cos t\sqrt{1+\sin t}$. Therefore, $dD=2\cos t\sqrt{1+\sin t}\cdot dt$. To find the damage caused between 0 hours and 6 hours you need to take a definite integral on this equation. Therefore, $D=\int_0^{1/4}2\cos t\sqrt{1+\sin t}\cdot dt$. This integral will require a u du substitution. If $u = 1 + \sin t$, then $du = \cos t \cdot dt$ and, if you take a 2 out of the integral, the substitution yields $D = 2\int_0^{1/4} u^{\frac{1}{2}} \cdot du$. Taking the integral results in the equation $D = \frac{4}{3}\Big|_0^{1/4} u^{\frac{3}{2}} = \frac{4}{3}\Big|_0^{1/4}(1+\sin t)^{\frac{3}{2}} = \frac{4}{3}\left(\left(1+\sin\left(\frac{1}{4}\right)\right)^{\frac{3}{2}} - (1+\sin(0))^{\frac{3}{2}}\right) \approx .524$. The damage done from the 6th to 12th hours is determined the same way except for a difference in bounds. $D = \frac{4}{3}\Big|_{1/4}^{1/2} u^{\frac{3}{2}} = \frac{4}{3}\Big|_{1/4}^{1/2}(1+\sin t)^{\frac{3}{2}} = \frac{4}{3}\left(\left(1+\sin\left(\frac{1}{2}\right)\right)^{\frac{3}{2}} - \left(1+\sin\left(\frac{1}{4}\right)\right)^{\frac{3}{2}}\right) \approx .542$. The rate at which the damage is occurring is increasing with time.

5) Evaluate $\int_1^5 \frac{x^2}{4x^3 - 1}dx$

Answer: Again, a u du substitution is necessary to complete the integral. If $u = 4x^3 - 1$, then $du = 12x^2 \cdot dx$. In order to substitute, we need a 12 inside the integral which means we need to compensate by putting a one-twelfth in front. Therefore, $\frac{1}{12}\int_1^5 \frac{12x^2}{4x^3-1}dx$ and substitution yields $\frac{1}{12}\int_1^5 \frac{du}{u}$. This is an impossible integral to take using the power rule which is why you learned a special rule for this situation. $\frac{1}{12}\int_1^5 \frac{du}{u} = \frac{1}{12}\Big|_1^5 \ln|u| = \frac{1}{12}(\ln(5) - \ln(1)) = \frac{\ln 5}{12}$

6) Find the equation of the function from the differential equation $\frac{dy}{dx} = \frac{-5x}{(2x^2-3)^4}$ if the function contains the point (1, 2).

Answer: Get dy alone which yields $dy = \frac{-5x}{(2x^2-3)^4}dx$. To solve for y we must integrate both sides to get $y = \int \frac{-5x}{(2x^2-3)^4}dx$. Again, use a u du substitution with $u = 2x^2 - 3$ so $du = 4x \cdot dx$. If you pull the –5 out of the integral and replace it with a 4, you will need to multiply the outside by one-fourth to get $y = \frac{-5}{4}\int \frac{4x}{(2x^2-3)^4}\,dx$. Substitution yields $y = \frac{-5}{4}\int u^{-4}\,du$ and taking the integral and replacing the x's gives you $y = \frac{5}{12}\left(2x^2-3\right)^{-3}+c$ which simplifies to $y = \frac{5}{12\left(2x^2-3\right)^3}+c$. Because you are told that the original function goes through the point (1, 2), c can be found! Substitution yields $2 = \frac{5}{12\left(2(1)^2-3\right)^3}+c \rightarrow$ $2 = \frac{5}{12\left(2(1)^2-3\right)^3}+c \rightarrow 2 = \frac{-5}{12}+c \rightarrow c = \frac{29}{12}$. This results in the final equation of $y = \frac{5}{12\left(2x^2-3\right)^3}+\frac{29}{12}$.