Teaching Notes for Calculus
Homework #18
Using Integration to Find Bounded Areas

*Review how the definite integral calculates the area between the bounds on the interval and between the curve and the axis. Also review how to rotate the graph and the integral when the graph is expressed as a function of y and mention the conditions when multiple bounds must be used. You will need to use these skills to complete classroom example #4 below. A quick review of both of these concepts will help students to better understand how to find the area between two curves.

Important notes: Always graph the curves first. If both curves are entirely above the x axis on the interval or one is entirely above while the other is totally below, and they do not intersect, simply subtract the integral of the lower function from the upper function. If the curves either intersect each other or intersect the x axis on the interval, you need to find these intersections as you will have to use the x values of these intersections as break points and create multiple integrals using these intersections as bounds. It is also possible that you will not be given any bounds for integration. In this case, the area you will be asked to find will be between two intersecting curves. You will have to solve the system of equations in order to find these bounds. There are multiple scenarios possible and each of the following classroom examples will help illustrate each one.

Classroom Examples

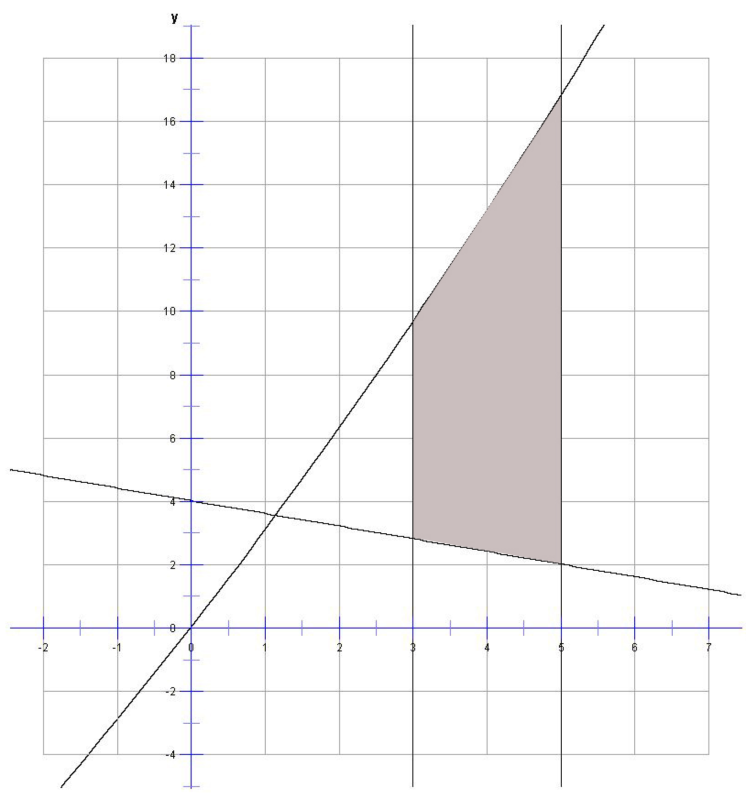
1) Graph the region bounded by the following equations, create a definite integral that gives the area of that region, and find the area:

$$y=\frac{x^2}{14}+3x, y=\frac{-2x}{5}+4, x=3, x=5$$

Answers:

$$\int_3^5\left(\frac{x^2}{14}+3x\right)dx-\int_3^5\left(\frac{-2x}{5}+4\right)dx\rightarrow$$

$$\left.\left(\frac{x^3}{42}+\frac{3x^2}{2}\right)\right|_3^5-\left.\left(\frac{-x^2}{5}+4x\right)\right|_3^5\rightarrow$$

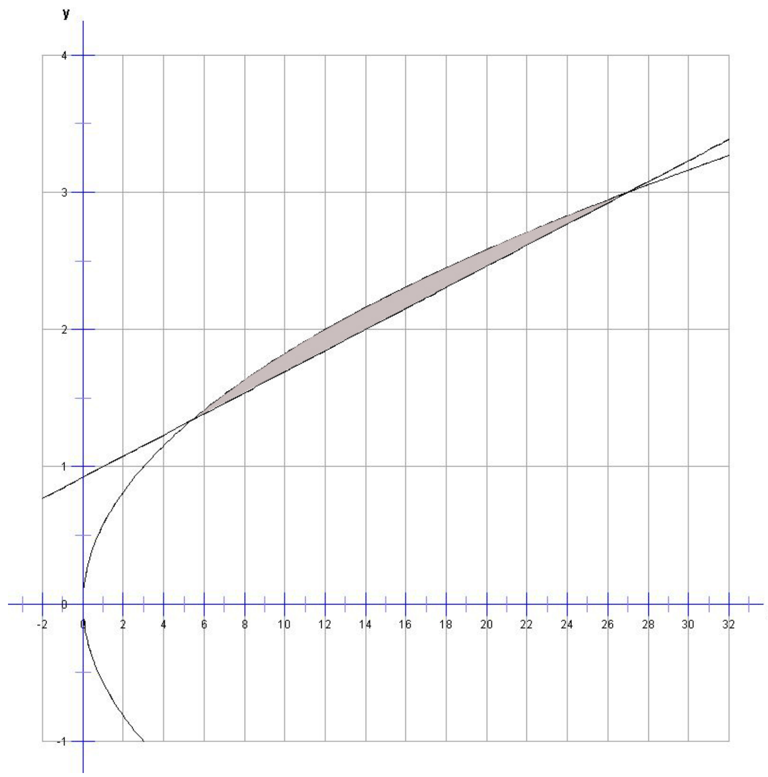


$$\left(\frac{125}{42}+\frac{75}{2}\right)-\left(\frac{9}{14}+\frac{27}{2}\right)-\Big|_{3}^{5}\left(\frac{-x^2}{5}+4x\right)\rightarrow\frac{79}{3}-\left(\left(-5+20\right)-\left(\frac{-9}{5}+12\right)\right)\rightarrow\frac{79}{3}-\frac{24}{5}\rightarrow$$

The area is $\frac{323}{15}$.

2) Graph the region bounded by the following equations, create a definite integral that gives the area of that region, and find the area:

$x=3y^2, x=13y-12$



Answers: In this case, the intersections of these two equations create the bounds for integration. Also, since both of these are functions of x, in terms of y, you must integrate along the y axis. Solve this system of equations by substitution and setting the results equal to zero. This gives you $3y^2-13y+12=0$. Factoring yields $(3y-4)(y-3)=0$ and solving yields $y=\frac{4}{3}$ or $y=3$. A quick look at the graph will confirm that these numbers can serve as the bounds for the integration on the y axis. Rotating the graph reveals that the linear equation is actually the upper equation and the parabola is the lower.

$$\int_{4/3}^{3}(13y-12)dy-\int_{4/3}^{3}\left(3y^2\right)dy\rightarrow\Big|_{4/3}^{3}\left(\frac{13y^2}{2}-12y\right)-\Big|_{4/3}^{3}\left(y^3\right)\rightarrow$$

$$\left(\frac{117}{2}-36\right)-\left(\frac{104}{9}-16\right)-\Big|_{4/3}^{3}\left(y^3\right)\rightarrow\frac{485}{18}-\left(\left(27\right)-\left(\frac{64}{27}\right)\right)\rightarrow$$

The area is $\frac{125}{54}$.

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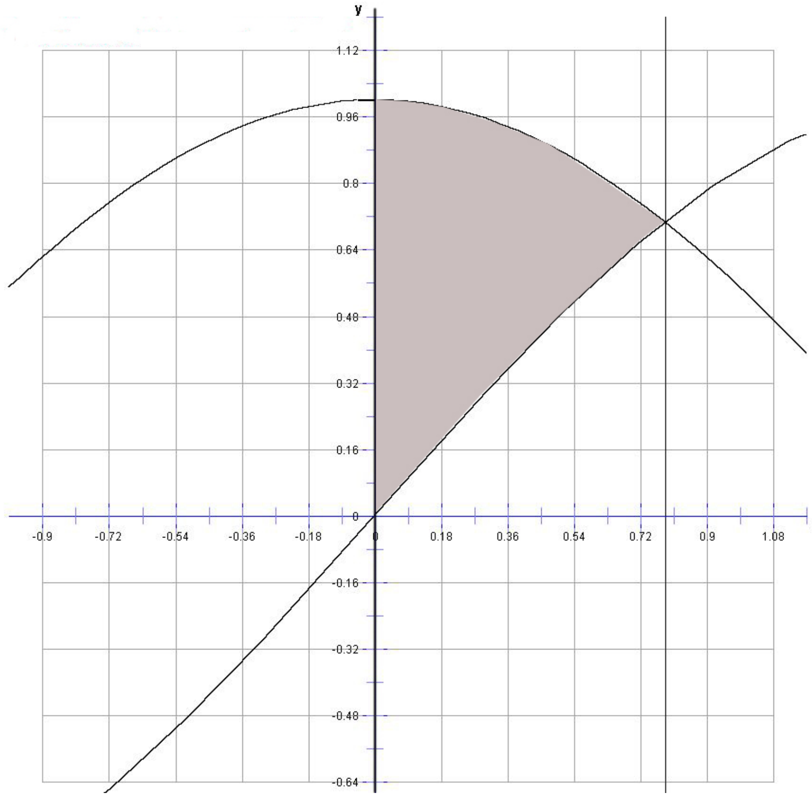
3) Graph the region bounded by the following equations, create a definite integral that gives the area of that region, and find the area:



$f(x) = \cos x, g(x) = \sin x, x = 0, x =$

Answers:

$$\int_0^{\pi/4} \cos(x)dx - \int_0^{\pi/4} \sin(x)dx \rightarrow$$

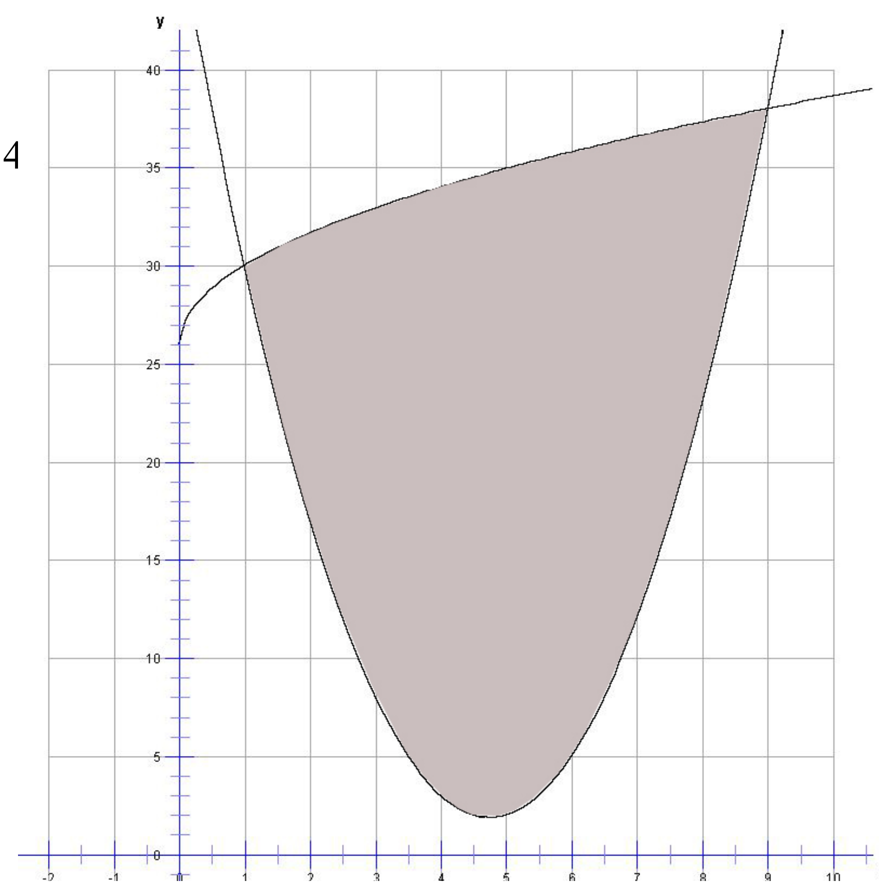
$$\Big|_0^{\pi/4} \sin(x) - \Big|_0^{\pi/4} -\cos(x) \rightarrow$$

$$\left(\frac{\sqrt{2}}{2} - 0\right) - \left(\frac{-\sqrt{2}}{2} + 1\right) \rightarrow$$

The area is $\sqrt{2} - 1$.

4) Graph the region bounded by

$y = 26 + \sqrt{16x}, y = 2x^2 - 19x + 4$. Create a definite integral with respect to x and then create a definite integral with respect to y that gives the area of that region, find the area using both integrals, and compare your results. Which method is simpler? In general, will this method always be simpler than the other one? Why or why not?



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Answers: To integrate, with respect to either the x or y, you need to find the intersections of these two equations. To solve, set them equal to each other to get the equation: $26+\sqrt{16x}=2x^2-19x+47$. Isolate the radical and square both sides to get $16x=\left(2x^2-19x+21\right)^2$ which simplifies to $16x=4x^4-76x^3+445x^2-798x+441$. Setting this equation equal to zero yields $0=4x^4-76x^3+445x^2-814x+441$. The list of possible rational roots for this equation is: $\pm 1,3,$
$7,9,21,49,63,103,147,441,\frac{1}{2},\frac{3}{2},\frac{7}{2},\frac{9}{2},\frac{21}{2},\frac{49}{2},\frac{63}{2},\frac{103}{2},\frac{147}{2},\frac{441}{2},\frac{1}{4},\frac{3}{4},\frac{7}{4},\frac{9}{4},\frac{21}{4},\frac{49}{4},\frac{63}{4},\frac{103}{4},\frac{147}{4},\frac{441}{4}$

. Synthetic division reveals +1 and +9 are the only rational roots. Therefore, the factored equation is $(x-1)(x-9)(4x^2-36x+49)=0$. Solving $4x^2-36x+49=0$, yields the two irrational answers of $x=\frac{36\pm 16\sqrt{2}}{8}$ which simplify to $x=\frac{9-4\sqrt{2}}{2}$ and $x=\frac{9+4\sqrt{2}}{2}$. Checking all four real answers in the original two equations reveals that only $x=1$ and $x=9$ are actual solutions. When $x=1$, substitution yields $y=30$. When $x=9$, substitution yields $y=38$. Therefore, the integral with respect to x is $\int_1^9\left(26+\sqrt{16x}\right)dx-\int_1^9\left(2x^2-19x+47\right)dx$. Integrating this equation yields

$$\Big|_1^9\left(26x+\frac{8\sqrt{x^3}}{3}\right)-\Big|_1^9\left(\frac{2x^3}{3}-\frac{19x^2}{2}+47x\right)\to\left(306-\frac{86}{3}\right)-\left(\frac{279}{2}-\frac{229}{6}\right)\to\frac{832}{3}-\frac{304}{3}.$$

Therefore, the area, by integrating with respect to the x axis, or using vertical rectangles, is 176.

Integrating with respect to the y axis requires that you solve both equations for x, use the solutions for y as the beginning and ending bounds, and then determine if you have to break up that interval if there is a change in either function over that interval. Solving the first equation yields $x=\frac{y^2}{16}-\frac{13y}{4}+\frac{169}{4}$ while solving the other requires completing the square and eventually yields $x=\dfrac{19\pm\sqrt{8y-15}}{4}$. Examining the rotated graph reveals that the upper and lower functions, between $y=\dfrac{15}{8}$ and $y=30$, are $x=\dfrac{19+\sqrt{8y-15}}{4}$ and $x=\dfrac{19-\sqrt{8y-15}}{4}$ while the upper and lower functions between $y=30$ and $y=38$, are $x=\dfrac{19+\sqrt{8y-15}}{4}$ and $x=\dfrac{y^2}{16}-\dfrac{13y}{4}+\dfrac{169}{4}$.

$$\int_{15/8}^{30}\left(\frac{19+\sqrt{8y-15}}{4}\right)dy-\int_{15/8}^{30}\left(\frac{19-\sqrt{8y-15}}{4}\right)dy+\int_{30}^{38}\left(\frac{y^2}{16}-\frac{13y}{4}+\frac{169}{4}\right)dy-\int_{30}^{38}\left(\frac{19+\sqrt{8y-15}}{4}\right)d$$

would represent the integral with respect to y. Since this integral is so large and complicated, we will break up the work and start by just focusing on the first boundary integrals. $\displaystyle\int_{15/8}^{30}\left(\frac{19}{4}+\frac{\sqrt{8y-15}}{4}\right)dy-\int_{15/8}^{30}\left(\frac{19}{4}-\frac{\sqrt{8y-15}}{4}\right)dy\to$

$$\Big|_{15/8}^{30}\left(\frac{19y}{4}+\frac{\sqrt{(8y-15)^3}}{48}\right)-\Big|_{15/8}^{30}\left(\frac{19y}{4}-\frac{\sqrt{(8y-15)^3}}{48}\right)\to\left(\left(\frac{285}{2}+\frac{1125}{16}\right)-\left(\frac{285}{32}+0\right)\right)-$$

$$\left(\left(\frac{285}{2}-\frac{1125}{16}\right)-\left(\frac{285}{32}-0\right)\right)\to\left(\frac{6525}{32}\right)-\left(\frac{2025}{32}\right)=\frac{1125}{8}.$$

We now have to evaluate the integral for the second boundary:

$$\int_{30}^{38}\left(\frac{19}{4}+\frac{\sqrt{8y-15}}{4}\right)dy-\int_{30}^{38}\left(\frac{y^2}{16}-\frac{13y}{4}+\frac{169}{4}\right)dy\to\Big|_{30}^{38}\left(\frac{19y}{4}+\frac{\sqrt{(8y-15)^3}}{48}\right)-$$

$$\Big|_{30}^{38}\left(\frac{y^3}{48}-\frac{13y^2}{8}+\frac{169y}{4}\right)\to$$

$$\left(\left(\frac{361}{2}+\frac{4913}{48}\right)-\left(\frac{285}{2}+\frac{1125}{16}\right)\right)-\left(\left(\frac{6859}{6}-\frac{4693}{2}+\frac{3211}{2}\right)-\left(\frac{1125}{2}-\frac{2925}{2}+\frac{2535}{2}\right)\right)\to$$

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$\left(\frac{1681}{24}\right)-\left(\frac{104}{3}\right)=\frac{283}{8}$. Combining the area found from the first bounds with those from the second bounds yields $\frac{1125}{8}+\frac{283}{8}=176$. Therefore, the area, by integrating with respect to the y axis, or using horizontal rectangles, is 176. This proves that the area in question is 176 regardless of the variable used.

While demonstrating the validity, and proving that the areas are equal no matter which variable you use for the integration, the most important lesson that the students should take from this is that their choice of whether to integrate with respect to x or y will, in general, determine the amount and complexity of the work involved in finding the area…In other words, choose wisely!!!