Teaching Notes for Calculus
Homework #19
Exponential Derivatives and Integrals and Using Partial Fractions to Evaluate Difficult Integrals

Review the following special derivative and integrals rules for exponentials and logarithms and make sure that the students know they made need to use them, especially for the homework problems for this lesson.

$$\frac{d}{dx}(a^x) = a^x \ln(a)$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, x > 0$$

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

$$\frac{d}{dx}(\log_a(x)) = \frac{1}{x\ln(a)}$$

$$\int a^x dx = \frac{1}{\ln a}a^x + c$$

$$\int \frac{1}{x}dx = \ln|x| + c$$

Also review the decomposition of a fraction into partial fractions from pre-calculus. Remember that you must use long division first if the largest power on top is greater than or equal to the largest on the bottom. Make sure that you REALLY check for the largest exponents, especially if the bottom is already factored. In that case, you need to think through what the largest power would be if you multiplied out all of the factors. For example, in the sample problem below, the largest exponent on the top is 6, but it isn't clear what the largest exponent on the bottom is. If you multiply out the bottom, the largest exponent is 7 so no long division is necessary. If you, or the students, need a refresher on this concept, use the following sample problem to see the proper setup and rules for partial fractions:

$$\frac{2x^6 - 4x^5 + 5x^4 - 3x^3 + x^2 + 3x}{(x-1)^3(x^2+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}$$

Remember that anything that resembles something like x – 1 requires you to just put a letter, like A, on the top of that fraction, if that x – 1 is raised to a power, you need every version of that term, from the highest degree, down to the 1st degree (see the example above). Any term containing an x^2, like x^2+1 requires that you put something like Dx + E on the top of that fraction. After you have written out the original problem and made it equal to all of the partial fractions you created, you then wipe out fractions and substitute for x to try to isolate certain variables…in many cases, you must solve systems of equations to find A, B, C, etc.

Classroom Examples

1) Evaluate $\int 3e^x\sqrt{2+5e^x}\,dx$

Answer: Use a u du substitution by letting $u=2+5e^x$. Therefore, $du=5e^x dx$ which would be perfect except for the 3 not being a 5 in the original integral. This can be fixed, however, by multiplying the inside of the integral by five-thirds while balancing that out by multiplying the outside of the integral by three-fifths. The new integral, $\frac{3}{5}\int 5e^x\sqrt{2+5e^x}\,dx$, can be replaced by $\frac{3}{5}\int u^{\frac{1}{2}}du$. Integrating yields $\frac{2u^{\frac{3}{2}}}{5}+c$ and substitution gives you $\frac{2\sqrt{\left(2+5e^x\right)^3}}{5}+c$.

2) Evaluate $\int -4x^2\left(3^{2x^3}\right)dx$

Answer: Again, use a u du substitution by letting $u=2x^3$. Therefore, $du=6x^2dx$ which would be perfect except for the –4 not being a 6 in the original integral. This can be fixed, however, by multiplying the inside of the integral by negative three-halves while balancing that out by multiplying the outside of the integral by negative two-thirds. The new integral, $\frac{-2}{3}\int 6x^2\left(3^{2x^3}\right)dx$, can be replaced by $\frac{-2}{3}\int 3^u\,du$. Integrating yields $\frac{-2\cdot 3^u}{3\ln(3)}+c$ and substitution gives you $\frac{-2\cdot 3^{2x^3}}{3\ln(3)}+c$.

Simplifying gives you $\frac{-2\cdot 9^{x^3}}{3\ln(3)}+c$

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3) Evaluate $\int \frac{13x-6}{3x^2-7x-6}dx$ using partial fractions.

Answer: Decomposing $\frac{13x-6}{3x^2-7x-6}$ is accomplished by factoring the bottom and then writing out $\frac{13x-6}{(3x+2)(x-3)} = \frac{A}{3x+2} + \frac{B}{x-3}$. Clearing fractions yields $13x-6 = A(x-3)+B(3x+2)$. Substituting 3 for $\boldsymbol{x}$ allows you to solve for $\boldsymbol{B}$ as you get $33 = B(11)$ which results in $\boldsymbol{B} = 3$. Knowing that $\boldsymbol{B} = 3$, you can make $\boldsymbol{x}$ anything but 3 to solve for $\boldsymbol{A}$. Therefore, making $\boldsymbol{x} = 0$ results in the equation $-6 = A(-3)+3(2)$. Solving for $\boldsymbol{A}$ gives you $\boldsymbol{A} = 4$. The original fraction is therefore decomposed into $\frac{4}{3x+2} + \frac{3}{x-3}$. This allows us to break up the original integration problem into two, workable integrals: $\int \frac{4}{3x+2}dx + \int \frac{3}{x-3}dx$. Both of these integrals can be evaluated using u du substitutions. For the first integral, let $u = 3x+2$ so $du = 3dx$ which would be perfect except for the 4 not being a 3 in the original integral. This can be fixed, however, by multiplying the inside of the integral by three-fourths while balancing that out by multiplying the outside of the integral by four-thirds. The new integral, $\frac{4}{3}\int \frac{3}{3x+2}dx$, can be replaced by $\frac{4}{3}\int \frac{1}{u}du$. Integrating yields $\frac{4\ln|u|}{3}+c$ and substitution gives you $\frac{4\ln|3x+2|}{3}+c$. For the second integral, let $u = x-3$ so $du = 1dx$ which would be perfect except for the 3 not being a 1 in the original integral. This can be fixed, however, by multiplying the inside of the integral by one-third while balancing that out by multiplying the outside of the integral by three. The new integral, $3\int \frac{1}{x-3}dx$, can be replaced by $3\int \frac{1}{u}du$. Integrating yields $3\ln|u|+c$ and substitution gives you $3\ln|x-3|+c$. Therefore, the final answer is $\frac{4\ln|3x+2|}{3} + 3\ln|x-3| + c$.

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4) Evaluate $\int \frac{6+3e^{2x}}{4e^{3x}}dx$

Answer: In this case, you can use basic algebra to factor out a one-fourth and split this problem up into the following two integrals: $\frac{1}{4}\int \frac{6}{e^{3x}}dx + \frac{1}{4}\int \frac{3e^{2x}}{e^{3x}}dx$ which simplifies to $\frac{1}{4}\int \frac{6}{e^{3x}}dx + \frac{1}{4}\int \frac{3}{e^{x}}dx$ and can be further simplified to $\frac{3}{2}\int 1e^{-3x}dx + \frac{3}{4}\int 1e^{-x}dx$. We can now employ a u du substitution on each integral. For the first integral, if $u = -3x$ then $du = -3dx$. This would be perfect except for the 1 not being a –3 in the original integral. This can be fixed, however, by multiplying the inside of the integral by negative three while balancing that out by multiplying the outside of the integral by negative one-third. The new integral, $\frac{-1}{2}\int -3e^{-3x}dx$, can be replaced by $\frac{-1}{2}\int e^{u}du$. Integrating yields $\frac{-1e^{u}}{2\ln(e)}+c$ which simplifies to $\frac{-1e^{u}}{2}+c$. Substitution gives you $\frac{-1e^{-3x}}{2}+c$. For the second integral, if $u = -1x$ then $du = -1dx$. This would be perfect except for the 1 not being a –1 in the original integral. This can be fixed, however, by multiplying the inside of the integral by negative one while balancing that out by multiplying the outside of the integral by negative one. The new integral, $-\frac{3}{4}\int -1e^{-x}dx$, can be replaced by $-\frac{3}{4}\int e^{u}du$. Integrating yields $-\frac{3e^{u}}{4\ln(e)}+c$ which simplifies to $-\frac{3e^{u}}{4}+c$. Substitution gives you $-\frac{3e^{-x}}{4}+c$.Therefore, the final answer is $\frac{-1e^{-3x}}{2}-\frac{3e^{-x}}{4}+c$ or $\frac{-1}{2e^{3x}}-\frac{3}{4e^{x}}+c$.

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5) Evaluate $\int \frac{60x^3 - 51x^2 + 26x - 13}{20x^2 - 17x + 3} dx$ using partial fractions.

Answer: Recognizing that the degree of the top is larger than the degree on the bottom, you use long division to get $3x + \frac{17x-13}{20x^2 - 17x + 3}$. Decomposing $\frac{17x-13}{20x^2 - 17x + 3}$ is accomplished by factoring the bottom and then writing out $\frac{17x-13}{20x^2 - 17x + 3} = \frac{A}{4x-1} + \frac{B}{5x-3}$. Clearing fractions yields $17x - 13 = A(5x-3) + B(4x-1)$. Substituting three-fifths for ***x*** allows you to solve for ***B*** as you get $\frac{-14}{5} = B(\frac{7}{5})$ which results in ***B*** = –2. Knowing that ***B*** = –2, you can make ***x*** anything but three-fifths to solve for ***A***. Therefore, making ***x*** = 0 results in the equation $-13 = A(-3) - 2(-1)$. Solving for ***A*** gives you ***A*** = 5. The original fraction is therefore decomposed into $\frac{5}{4x-1} - \frac{2}{5x-3}$. This allows us to break up the original integration problem into two, workable integrals: $\int \frac{5}{4x-1} dx + \int \frac{-2}{5x-3} dx$. Both of these integrals can be evaluated using u du substitutions. For the first integral, let $u = 4x - 1$ so $du = 4dx$ which would be perfect except for the 5 not being a 4 in the original integral. This can be fixed, however, by multiplying the inside of the integral by four-fifths while balancing that out by multiplying the outside of the integral by five-fourths. The new integral, $\frac{5}{4}\int \frac{4}{4x-1} dx$, can be replaced by $\frac{5}{4}\int \frac{1}{u} du$. Integrating yields $\frac{5\ln|u|}{4} + c$ and substitution gives you $\frac{5\ln|4x-1|}{4} + c$. For the second integral, let $u = 5x - 3$ so $du = 5dx$ which would be perfect except for the –2 not being a 5 in the original integral. This can be fixed, however, by multiplying the inside of the integral by negative five-halves while balancing that out by multiplying the outside of the integral by negative two-fifths. The new integral, $\frac{-2}{5}\int \frac{5}{5x-3} dx$, can be replaced by $\frac{-2}{5}\int \frac{1}{u}\,du$. Integrating yields $\frac{-2\ln|u|}{5}+c$ and substitution gives you $\frac{-2\ln|x-3|}{5}+c$. Therefore, the final answer is $\frac{5\ln|4x-1|}{4}+\frac{-2\ln|x-3|}{5}+c$.