Teaching Notes for Calculus
Homework #20
The Disk and Washer Methods for Volumes of Revolutions

To truly understand this concept, you need to remember the general relationship between length, area, and volume. Take the relationship between a line, a square, and a cube. If the line has length x, then the area of the square would be x^2, and the volume of the cube would be x^3. You could also examine the relationship between a line like the radius, a circle, and a sphere. If the line has length r, then the area of the circle would be πr^2, and the volume of the sphere would be $\frac{4\pi r^3}{3}$. Notice that in both of these examples the areas are based on increasing the exponent on the length by one. Likewise, the volumes are based on increasing the exponent on the length, in the area formula, by one. The concept we are currently covering, integration, just happens to involve a power rule which increases the exponent on the variable by one. As we've recently discovered, integrating an equation of a curve (like a line), results in the area between the axis and the curve. Therefore, it should follow that integrating the area between the axis and the curve, as you rotated that curve about the axis, would result in the volume of that shape. In fact, it does find the volume of that revolved area and is formally called the volume of revolution. Since we will be rotating the curve 360 degrees about the axis, then each infinitesimally small slice, or disk, would basically be a circle. The area of a circle, as shown above, is πr^2, but the radius is changing with the curve, r, so that the radius must be defined as a function of x, or $R(x)$. As shown earlier, integrating the area would result in the volume; which, in this case is the volume of revolution about the axis. Since you are finding the volume using infinitesimally thin disks, this method is called the disk method. You can also revolve your curve about either axis. If you rotate it about the x-axis, known as a horizontal volume of revolution, the formula would be $V = \pi\int_a^b [R(x)]^2\,dx$. If you rotate it about the y-axis, known as a vertical volume of revolution, the formula would be $V = \pi\int_a^b [R(y)]^2\,dy$.

On Worksheet 18, the students learned that they could not only find the area between the axis and the curve, but that they could also find the area between two curves. If this is the case, and you wanted to find the volume of revolution for the shape formed by rotating both curves about the axis, you would be creating infinitesimally thin washers (disks with holes in the middle, like a donut). In this type of situation, since you are finding the volume using infinitely thin washers, this method is known as the washer method. If you rotate about the x-axis, or a horizontal volume of revolution, the formula would be $V = \pi\int_a^b \left([R(x)]^2 - [r(x)]^2\right)dx$. If you rotate it about the y-axis, or a vertical volume of revolution, the formula would be $V = \pi\int_a^b \left([R(y)]^2 - [r(y)]^2\right)dy$.

Important Note: When revolving over a line that is not an axis, you MUST subtract the function furthest away from the line of revolution from the number value of that line BEFORE you square it. If there are two functions, you must do the same thing for the equation closest to the line of revolution and then subtract the square of that from the answer from the square of the equation that was furthest away! This might sound a bit confusing, but it should make MUCH more sense after you review example #3 and #5.

Helpful resource: As of this writing, there is a website that helps you to visualize your volumes of revolutions. The website http://www.shodor.org/interactivate/activities/FunctionRevolution/ is a very user-friendly application for graphing volumes of revolution. The program even allows you to rotate the final 3-D graph in any direction you choose!

Classroom Examples

1) Find the volume of the solid generated by revolving the region bounded by the equations $y = x, x = 6, y = 0$ about the x-axis.

Answer: Students should always make a quick sketch so they can visualize the bounded region and then use their imaginations to try to "see" what the revolution of that region would resemble as a solid. In this case, the sketch would resemble a right triangle along the x-axis with the vertex of the right angle located at the point $(6,0)$ and vertex of one of the other angles at the point $(0,0)$. Rotating this right triangle about the x-axis would produce a solid with the shape of a right, circular cone with a radius of 6 and a height of 6. Use the formula $V = \pi\int_a^b [R(x)]^2\,dx$, where $R(x) = x, a = 0, b = 6$. Therefore, the volume would be given by $V=\pi\int_0^6 [x]^2\,dx=\pi\Big|_0^6\frac{x^3}{3}=\pi(72-0)=72\pi$. This example provides the students with an excellent opportunity to "prove" that this process actually works. Remembering that the solid resembled a right, circular cone with a radius of 6 and a height of 6, and using the formula $V=\frac{1}{3}\pi r^2 h$ which, from geometry, yields the same answer of 72π.

2) Find the volume of the solid generated by revolving the region bounded by the equations $x=\sqrt{64-(y-8)^2}$ and $x=0$ about the y-axis.

Answer: Students should always make a quick sketch so they can visualize the bounded region and then use their imaginations to try to "see" what the revolution of that region would resemble as a solid. In this case, the sketch would resemble a semicircle along the y-axis with the center being $(0,8)$ and the radius of the semicircle being 8. Rotating this semicircular region about the y-axis would produce a solid with the shape of a sphere of radius of 8. Use the formula $V=\pi\int_a^b [R(y)]^2\,dy$, where $R(x)=\sqrt{64-(y-8)^2}, a=0, b=16$.
Therefore, the volume would be given by

$$V=\pi\int_0^{16}\left[\sqrt{64-(y-8)^2}\right]^2 dy=\pi\Big|_0^{16}\left(64y-\frac{(y-8)^3}{3}\right)=\left(1024\pi-\frac{512\pi}{3}\right)-\left(0+\frac{512\pi}{3}\right)=\frac{2048\pi}{3}.$$

Again, this example provides the students with an excellent opportunity to "prove" that this process actually works. Remembering that the solid resembled a sphere with a radius of 8 and using the formula $V=\frac{4}{3}\pi r^3$ which, from geometry, yields the same answer of $\frac{2048\pi}{3}$.

3) Find the volume of the solid generated by revolving the region bounded by the equations $R(x)=5x$, $r(x)=2x$, $x=2$, and $x=6$ about the line $y=4$.

Answer: Once again, students should always make a quick sketch so they can visualize the bounded region and then use their imaginations to try to "see" what the revolution of that region would resemble as a solid. In this case, the sketch would resemble a trapezoid with the two parallel sides being at $x=2$, and $x=6$. Rotating this trapezoid about the line $y=4$ would produce a solid with the shape of a truncated cone with a cone-shaped hole down the center which increases in size as the cone gets larger. Since you are revolving about a line that resembles the x axis, but shifted up, we must us the formula for rotating about the x axis, a horizontal revolution, using the washer method version: $V=\pi\int_a^b\left(\left[R(x)\right]^2-\left[r(x)\right]^2\right)dx$. Remember that, when rotating around a line that is neither the y or x axes, you MUST subtract the function furthest away from the line of revolution from the number value of that line BEFORE you square it. If there are two functions, as in this case, you must do the same thing for the equation closest to the line of revolution and then subtract the square of that from the answer from the square of the equation that was furthest away. In this case, the function furthest away from the line of rotation is $R(x)=5x$. Therefore, the volume would be given by $V=\pi\int_2^6\left(\left[4-5x\right]^2-\left[4-2x\right]^2\right)dx$ and can be expanded to yield

$$V=\pi\int_2^6\left(\left(16-40x+25x^2\right)-\left(16-16x+4x^2\right)\right)dx=\pi\int_2^6\left(21x^2-24x\right)dx=\pi\Big|_2^6\left(7x^3-12x^2\right)=$$
$$\pi\Big|_2^6\left(7x^3-12x^2\right)=\pi\left(1286-16\right)=1072\pi.$$

4) Find the volume of the solid generated by revolving the region bounded by the equations $y=\sqrt{\sin x}, x=0, y=0, y=\pi$ about the x-axis.

Answer: Make a quick sketch to visualize the bounded region and then use your imagination to try to "see" what the revolution of that region would resemble as a solid. In this case, the sketch resembles half of an ellipse with the straight edge along the x-axis and the revolving this region about the x axis would create a "footballish" shaped solid. Use the formula $V=\pi\int_a^b\left[R(x)\right]^2dx$, where $R(x)=\sqrt{\sin x}, a=0, b=\pi$. Therefore, the volume would be given by $V=\pi\int_0^6\left[\sqrt{\sin x}\right]^2dx=-\pi\Big|_0^\pi\cos x=-\pi\left(-1-1\right)=2\pi$.

5) Find the volume of the solid generated by revolving the region bounded by the equations $x=2-y^2$ and $x=1$ about the line $x=1$.

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Answer: Make a quick sketch so they can visualize the bounded region and then use your imagination to try to "see" what the revolution of that region would resemble as a solid. In this case, the sketch would resemble a "sideways" parabola with a flat, vertical side at $x = 1$. Rotating this region would produce another "footballish" shaped solid that's been squashed on both ends. To find the bounds for this region, you need to find the intersections of the two given equations. Substitution and basic algebraic manipulation reveals that the equations intersect at the points $(1,-1)$ and $(1,1)$ which means that the bounds would be $y = -1$ and $y = 1$. Since you are revolving about a line that resembles the y axis, we must us the formula for rotating about the y axis, a vertical revolution, using the washer method version: $V = \pi\int_a^b\left(\left[R(y)\right]^2 - \left[r(y)\right]^2\right)dy$. Remember that, when rotating around a line that is neither the y or x axes, you MUST subtract the function furthest away from the line of revolution from the number value of that line BEFORE you square it. If there are two functions, as in this case, you must do the same thing for the equation closest to the line of revolution and then subtract the square of that from the answer from the square of the equation that was furthest away. In this case, the function furthest away from the line of rotation is $R(y) = 2 - y^2$. Therefore, the volume would be given by $V = \pi\int_{-1}^{1}\left(\left[1-\left(2-y^2\right)\right]^2 - \left[1-(1)\right]^2\right)dy$. Since the line of rotation is the same as the equation closest to that line, this problem will simplify rapidly but must still be constructed correctly. The integral can be expanded to yield

$$V = \pi\int_{-1}^{1}\left(\left(1-2y^2+y^4\right)-(0)\right)dy = \pi\int_{-1}^{1}\left(y^4-2y^2+1\right)dy =$$
$$\pi\Big|_{-1}^{1}\left(\frac{y^5}{5}-\frac{2y^3}{3}+1y\right) = \pi\left[\left(\frac{1}{5}-\frac{2}{3}+1\right)-\left(\frac{-1}{5}+\frac{2}{3}-1\right)\right] = \pi\left[\left(\frac{8}{15}\right)-\left(\frac{-8}{15}\right)\right] = \frac{16\pi}{15}.$$

6) Find the volume of the solid generated by revolving the region bounded by the equations $y = \sqrt{x}$ and $y = x^2$ about the x-axis.

Answer: As always, make a quick sketch to visualize the bounded region and then use your imagination to try to "see" what the revolution of that region would resemble as a solid. In this case, the sketch would resemble a "torpedo" crashing into the origin at a 45 degree angle. To find the bounds for this rotation, you need to solve the given system of equations. Substitution and basic algebraic manipulations reveal that these equations intersect at the points $(0,0)$ and $(1,1)$ which means that the bounds would be $x=0$ and $x=1$. Since you are revolving two equations about the x axis, we must us the formula for rotating about the x axis, a horizontal revolution, using the washer method version: $V=\pi\int_a^b\left(\left[R(x)\right]^2-\left[r(x)\right]^2\right)dx$. Rotating this "torpedo" shaped region about the x-axis would produce a solid with the shape of a "weird," hollow cone with both a curved outer shell and curved inner shell. Substituting into this integral yields

$$V=\pi\int_0^1\left(\left[\sqrt{x}\right]^2-\left[x^2\right]^2\right)dx=\pi\int_0^1\left(x-x^4\right)dx=\pi\Big|_0^1\left(\frac{x^2}{2}-\frac{x^5}{5}\right)=\pi\left[\left(\frac{1}{2}-\frac{1}{5}\right)-(0)\right]=\frac{3\pi}{10}$$

7) Find the volume of the solid generated by revolving the region bounded by the equations $y=x^2+1, y=0, x=0,$ and $x=1$ about the y-axis.

Answer: Make a quick sketch so you can visualize the bounded region and then use your imagination to try to "see" what the revolution of that region would resemble as a solid. In this case, the sketch would resemble a vertical rectangle with a parabolic section taken out of the top. The left side of the rectangle, the shorter side is on the y-axis while the longer side is on the line $x=1$. The bottom of the rectangle lies on the x-axis while the top is marked by the intersection of the rectangle and the removed parabolic section. On the left, the intersection occurs at the point $(0,1)$ and the intersection on the right occurs at the point $(0,2)$. Rotating this region about the y-axis would produce a solid with the shape of a vertical cylinder with a parabolic "scoop" taken out of the top. This particular problem must be completed in two parts as the solid consists of two distinct solids if rotated about the y-axis. From $y=0$ to $y=1$ the solid is a cylinder. From $y=1$ to $y=2$ the solid is a cylindrical shell with a parabolic "scoop" removed. Therefore, use two of the volume formulas added together as follows:

$V=\pi\int_a^b\left[R(y)\right]^2dy+\pi\int_c^d\left(\left[R(y)\right]^2-\left[r(y)\right]^2\right)dy$, where $R(y)=1$ when $a=0$ and $b=1$ and $R(y)=1$ and $r(y)=\sqrt{y-1}$ when $c=1$ and $d=2$. Therefore, the volume would be given

by $V=\pi\int_0^1[1]^2\,dy+\pi\int_1^2\left([1]^2-\left[\sqrt{y-1}\right]^2\right)dy=\pi\int_0^1 1dy+\pi\int_1^2(2-y)\,dy=$
$\pi\Big|_0^1(y)+\pi\Big|_1^2\left(2y-\frac{y^2}{2}\right)^1=\pi(1-0)+\pi\left(2-\frac{3}{2}\right)=\frac{3\pi}{2}$.