Teaching Notes for Calculus
Homework #21
The Cylindrical Shell Method for Volumes of Revolutions and Integration by Parts

The last lesson introduced the concept of finding a volume of revolution using infinitesimally thin disks and washers. Several homework problems asked the students to find the volume of revolution in two different ways: 1) Using *dx* and *f(x)* and 2) Using *dy* and *f(y)*.

As you hopefully noticed, from the results of the last homework, in most cases, it is far easier to use one way over the other, but not both. In many circumstances it is actually impossible, or VERY difficult to use one of the ways. Therefore, you must have an alternative method for finding volumes of revolution when the disk or washer methods don't work. Your alternative is to use the **Cylindrical Shell Method**. This method allows you to create infinitesimally thin cylindrical shells as opposed to disks or washers. How do you know when to use this method? When you are asked to find a volume of revolution about the y-axis but you can't, or it would be extremely difficult to find *f(y)*. It allows you to calculate the volume of a vertical revolution when you can easily find *f(x)*. Inversely, if you are asked to find a volume of revolution about the x-axis but you can't, or it would be extremely difficult to find *f(x)*. It allows you to calculate the volume of a horizontal revolution when you can easily find *f(y)*. For vertical revolutions, the formula would be $V = 2\pi \int_a^b x \cdot f(x)dx$ where the bounds of integration, *a* and *b*, are on the x-axis and *a* needs to be greater than or equal to 0. For horizontal revolutions, the formula would be $V = 2\pi \int_a^b y \cdot f(y)dy$ where the bounds of integration, *a* and *b*, are on the y-axis and *a* needs to be greater than or equal to 0.

The Shell Method formulas make conceptual sense in that $2\pi x$ or $2\pi y$ represent the circumference of the shell and the *f(x)* or *f(y)* represent the height of the cylindrical shell.

To help students remember what to do when, explain that, when the variable used for integration matches the axis of revolution, they should use the disk or washer methods. When the variable used for the integration is the opposite of the axis of revolution, they should use the shell method.

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When the shell is bounded by two equations, you must subtract the lower equation from the upper as demonstrated in the following two equations:

$$V = 2\pi\int_a^b x(f(x)_{upper} - f(x)_{lower})dx \quad \text{or} \quad V = 2\pi\int_a^b y(f(y)_{upper} - f(y)_{lower})dy$$

When revolving over a line that is not the y-axis, like over the line $x = 4$, you MUST subtract *y* from the 4, but you do NOT do that for the *f(y)* equation. Likewise, when revolving over a line that is not the x-axis, like over the line $y = 3$, you MUST subtract *x* from the 3, but you do NOT do that for the *f(x)* equation. For a more detailed explanation, see the classroom examples

The other major concept on this worksheet is the Integration by Parts Method. This method can be tried if you need to integrate a problem which can't be integrated with any of the standard rules. Integration by Parts is basically the opposite, and is derived from, the Produce Rule for derivatives.

The formula is: $\int u\,dv = u \cdot v - \int v\,du$. This can be easily proved by the following logic: Begin with $y = u \cdot v$ where *u* and *v* are both functions of *x*. Taking the derivative of *y* requires the Product Rule which gives you $dy = u \cdot dv + v \cdot du$. Now, if you which to find the original *y*, you could take the integral of *dy*. This would yield $y = \int dy = \int\left(u \cdot dv + v \cdot du\right) = \int u \cdot dv + \int v \cdot du$. Knowing that *y* equals two different expressions gives you $\int u \cdot dv + \int v \cdot du = y = u \cdot v$. Manipulating this new equation yields $\int u \cdot dv + \int v \cdot du = u \cdot v \rightarrow \int u \cdot dv = u \cdot v - \int v \cdot du$, which is the formula for Integration by Parts.

*Helpful hints: You get to decide which part of the integral is *u* and which part is *dv*. A useful rule to follow is to choose *dv* so that it represents the most complicated portion of the integral that you can actually do an integral on and let *u* be something where *du* is simpler than u. Also, if the integral consists of a single term, you can still try to use Integration by Parts by choosing dv to be dx. For a more in-depth understanding of this concept, see the classroom examples #3 and #4.

Classroom Examples

1) Find the volume of the solid of revolution formed by revolving the region bounded by $y = x - x^3$, $y = 0, x = 0$, and $x = 1$ about the y-axis.

Answer: The question states that we must revolve about the y-axis, so the first step is to decide what method to use, washer/disk or shell. In this case, as we must revolve about the y-axis, if you can easily solve for x, then the washer/disk method would be appropriate. If you can easily solve for y, then the shell method would be appropriate. Not only is the equation already solved for y, or *f(x)*, it is impossible to get x alone! Therefore, we MUST use the shell method. Just as you did for worksheet #20, it is always advisable to make a sketch of the region and the bounds so that you feel confident in setting up the volume formula. In this case, the region is bounded by $y = x - x^3$ on the top and $y = 0$ on the bottom. Solving this system of equations yields intersections at $(0,0)$ and $(1,0)$. The formula to determine the volume of revolution would be

$$V = 2\pi\int_0^1 x\cdot\left(x - x^3\right)dx = 2\pi\int_0^1\left(x^2 - x^4\right)dx = 2\pi\Big|_0^1\left(\frac{x^3}{3} - \frac{x^5}{5}\right) = 2\pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{4\pi}{15}$$

2) Find the volume of the solid of revolution formed by revolving the region bounded by $x = e^{-y^2}$, $y = 0, x = 0$, and $y = 1$ about the x-axis.

Answer: The question states that we must revolve about the x-axis, so the first step is to decide what method to use, washer/disk or shell. In this case, as we must revolve about the x-axis, if you can easily solve for y, then the washer/disk method would be appropriate. If you can easily solve for x, then the shell method would be appropriate. Not only is the equation already solved for x, or *f(y)*, it is very difficult to get y alone and, even if you managed to get y alone, the mess you would be left with would be a nightmare to integrate! Therefore, it is foolish to not use the shell method. Just as you did for worksheet #20, it is always advisable to make a sketch of the region and the bounds so that you feel confident in setting up the volume formula. In this case, the region is bounded by $x = e^{-y^2}$ on the right, $x = 0$ on the left, and $y = 1$ on the top. Solving this system of equations yields intersections at $(1,0)$ and $\left(\frac{1}{e},1\right)$. The formula to determine the volume of revolution would therefore be $V = 2\pi\int_0^1 y\cdot\left(e^{-y^2}\right)dy$. To determine this integral will require a *u du* substitution. If $u = -y^2$, then $du = -2ydy$. If you multiply what's inside the integral by negative two and the outside by negative one-half, the volume equation would be $V = -1\pi\int_0^1 -2y\cdot\left(e^{-y^2}\right)dy$. Substitution yields the equation $V = -1\pi\int_0^1 e^u du = -1\pi\Big|_0^1 e^u = -1\pi\Big|_0^1 e^{-y^2} = -1\pi\left(\frac{1}{e}-1\right) = \frac{-\pi+\pi e}{e}$

3) Find the volume of the solid of revolution formed by revolving the region bounded by $y = 15 - x^2 + 2x$, $y = 0, x = 0$, and $x = 5$ about the line $x = 5$.

Answer: The question states that we must revolve about the line $x = 5$, so the first step is to decide what method to use, washer/disk or shell. In this case, as we must revolve about the line $x = 5$, a shift off of the y-axis, if you can easily solve for *x*, then the washer/disk method would be appropriate. If you can easily solve for *y*, then the shell method would be appropriate. Not only is the equation already solved for *y*, or *f(x)*, it is both time consuming and confusing to get *x* alone. Therefore, we should use the shell method. Just as you did for worksheet #20, it is always advisable to make a sketch of the region and the bounds so that you feel confident in setting up the volume formula. In this case, the region is bounded by $y = 15 - x^2 + 2x$ on the top and right, $x = 0$ on the left, and $y = 0$ on the bottom. Solving this system of equations yields intersections at $(0,15)$, $(0,0)$, and $(5,0)$. The formula to determine the volume of revolution would be $V = 2\pi\int_0^5 (5-x)\cdot\left(15-x^2+2x\right)dx$ because of the shift of 5 from the y-axis. Remember that you only use the 5 – shift on the part of the formula that is just *x*. You DO NOT shift the other equation. Expanding yields $V = 2\pi\int_0^5 \left(x^3 - 7x^2 - 5x + 75\right)dx$. Taking the integral yields $V = 2\pi\Big|_0^5\left(\frac{x^4}{4} - \frac{7x^3}{3} - \frac{5x^2}{2} + 75x\right) = 2\pi\left(\frac{625}{4} - \frac{875}{3} - \frac{125}{2} + 375\right) = \frac{2125\pi}{6}$

4) Find the volume of the solid of revolution formed by revolving the region bounded by $8x = y^2 - 28y + 320$ and $4x = y^2 - 28y + 208$ about the line $y = 30$.

Answer: The question states that we must revolve about the line $y = 30$, so the first step is to decide what method to use, washer/disk or shell. In this case, as we must revolve about the line $y = 30$, a shift off of the x-axis, if you can easily solve for *y*, then the washer/disk method would be appropriate. If you can easily solve for *x*, then the shell method would be appropriate. Not only are both equations easily solved for *x*, or *f(y)*, it is very difficult to get *y* alone and, even if you managed to get *y* alone, the mess you would be left with would be a nightmare to integrate! Therefore, it would be prudent to use the shell method. Just as the students did for the homework on worksheet #20, it is always advisable to make a sketch of the region and the bounds so that they feel confident in setting up the volume formula. In this case, the region is bounded by $x = \frac{y^2}{8} - \frac{7y}{2} + 40$ on the right and $x = \frac{y^2}{4} - 7y + 52$ on the left. Solving this system of equations yields intersections at $(28,4)$, and $(28,24)$. The formula to determine the volume of revolution would therefore need to include both upper and lower equations and just the y term needs to be shifted by 30. Therefore, the formula for the volume of revolution would be $V = 2\pi\int_4^{24}\left[(30 - y)\left(\left(\frac{y^2}{8} - \frac{7y}{2} + 40\right) - \left(\frac{y^2}{4} - 7y + 52\right)\right)\right]dy$. Due to the nature of the equations in this integral, it would be helpful to simplify and expand before we integrate. Simplification and expansion yields $V = 2\pi\int_4^{24}\left(\frac{y^3}{8} - \frac{29y^2}{4} + 117y - 360\right)dy$.

Integrating yields the equation

$$V = 2\pi\Big|_4^{24}\left(\frac{y^4}{32} - \frac{29y^3}{12} + \frac{117y^2}{2} - 360y\right) = 2\pi\left(2016 - \left(\frac{1952}{3}\right)\right) = \frac{16000\pi}{3}.$$

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5) Evaluate $\int xe^x dx$ using integration by parts if necessary.

Answer: After several attempts at trying to evaluate the integral using previously learned methods, you realize that it is impossible to find this integral without using integration by parts. Remembering the rules for this method, you should choose *dv* so that it represents the most complicated portion of the integral that you can actually do an integral on and let *u* be something where *du* is simpler than *u* in the formula $\int u\,dv = u \cdot v - \int v\,du$. In this case, you should make $u = x$ because that would make $du = 1dx$ which is simpler than *u*. If you tried to make $u = e^x$, then $du = e^x dx$ which is not any simpler. Therefore, $dv = e^x dx$, as long as you can integrate *dv*. Since the integral of $dv = e^x dx$ is $v = e^x$, we have found the proper substitutions. Substituting yields $\int udv = u \cdot v - \int vdu =$
$\int xe^x dx = x \cdot e^x - \int 1e^x dx = xe^x - e^x + c$

6) Evaluate $\int x^2 \sin xdx$ using integration by parts if necessary.

Answer: After several attempts at trying to evaluate the integral using previously learned methods, you realize that it is impossible to find this integral without using integration by parts. Remembering the rules for this method, you should choose *dv* so that it represents the most complicated portion of the integral that you can actually do an integral on and let *u* be something where *du* is simpler than *u* in the formula $\int u\,dv = u \cdot v - \int v\,du$. In this case, you should make $u = x^2$ because that would make $du = 2xdx$ which is simpler than *u*. If you tried to make $u = \sin x$, then $du = \cos xdx$ which is not any simpler. Therefore, $dv = \sin xdx$, as long as you can integrate *dv*. Since the integral of $dv = \sin xdx$ is $v = -\cos x$, we have found the proper substitutions. Substituting yields
$\int udv = u \cdot v - \int vdu = \int x^2 \sin xdx = -x^2 \cdot \cos x - \int -2x \cos xdx = -x^2 \cdot \cos x + 2\int x \cos xdx$.
The problem now is that you still can't integrate $x \cos xdx$. Luckily, you can use integration by parts as many times as needed, as long the integral continually simplifies. Therefore, take $\int x \cos xdx$ and evaluate just that using integration by parts. Using the same logic as before, make $u = x$, which makes $du = 1dx$. Likewise, making $dv = \cos xdx$ makes $v = \sin x$. Substituting yields $\int udv = u \cdot v - \int vdu =$

$\int x \cos x dx = x \cdot \sin x - \int \sin x dx =$ $x \cdot \sin x + \cos x + c$. Finally, substitute $x \cdot \sin x + \cos x + c$ in for $\int xcoxdx$ in the original answer to get $-x^2 \cdot \cos x + 2(x \cdot \sin x + \cos x + c)$. Simplifying yields the final answer of $-x^2 \cdot \cos x + 2x \sin x + 2\cos x + c \cdot$

7) Evaluate $\int \sec^3 x dx$ using integration by parts if necessary.

Answer: After several attempts at trying to evaluate the integral using previously learned methods, you realize that it is impossible to find this integral without using integration by parts. The issue with this particular integral is that it appears to only have one part. In this case, you have two options. The first option, as mentioned in the teaching notes, is to use the *dx* as its own part. I quick analysis, however, reveals that you still end up with an impossible integral. The second option, which only applies in specific circumstances, is that, if the single part can be easily broken up into two parts, you should attempt to separate the term and hope that things work out. The second option applies to this problem as you can break up $\sec^3 x$ into $\sec x$ and $\sec^2 x$. Remembering the rules for this method, you should choose *dv* so that it represents the most complicated portion of the integral that you can actually do an integral on and let *u* be something where *du* is simpler than *u* in the formula $\int u\,dv = u \cdot v - \int v\,du$. In this case, you should make $dv = \sec^2 xdx$ because that's the most complicated part and it is easily integrated to give you $v = \tan x$. This forces you to make $u = \sec x$ which, after taking the derivative, yields $du = \sec x \cdot \tan xdx$. While this is NOT simpler than *u*, it represents the most feasible option. If this doesn't work, you will have to go back and switch your choices.
Substituting yields $\int udv = u \cdot v - \int vdu = \int \sec x \cdot \sec^2 xdx = \sec x \cdot \tan x - \int \tan^2 x \cdot \sec xdx$. The new integral is complicated enough to make you second guess your substitutions. However, if you use a trig substitution, there may still be hope. Substituting $\sec^2 x - 1$ for $tan^2 x$ in the new integral yields $\int (\sec^2 x - 1) \cdot \sec xdx \to \int (\sec^3 x - \sec x) dx \to \int \sec^3 xdx - \int \sec xdx$. Substituting this result back into our original answer yields $\int \sec^3 xdx = \sec x \cdot \tan x - \left(\int \sec^3 xdx - \int \sec xdx\right)$. Simplifying this equation yields

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$\int \sec^3 x dx = \sec x \cdot \tan x - \int \sec^3 x dx + \int \sec x dx$. Adding $\int \sec^3 x dx$ to both sides yields $2\int \sec^3 x dx = \sec x \cdot \tan x + \int \sec x dx$. This clever mathematical maneuver leaves us with a simple, straightforward integral to evaluate. Using one of the integral trig rules, $\int \sec x dx = \ln\left|\sec x + \tan x\right| + c$, results in the equation $2\int \sec^3 x dx = \sec x \cdot \tan x + \ln\left|\sec x + \tan x\right| + c$. Now, to find the final answer, all we need to do is divide by 2. The final answer is $\int \sec^3 x dx = \frac{\sec x \cdot \tan x}{2} + \frac{\ln\left|\sec x + \tan x\right|}{2} + c$.