Teaching Notes for Calculus
Homework #22
Mass, Center of Mass, and Moments Involving Integration

Using integration, it is possible to find the **Mass** of a planar lamina (thin sheet of uniform density) given upper bounds f(x) and lower bounds g(x)

The universal symbol for uniform density is ρ and this means that the mass per unit volume doesn't vary. This leads us to an important integral equation for mass:

$$mass = density(area) \therefore mass \text{ or } m = \rho\int_a^b [f(x) - g(x)]dx$$

Moment is just like torque but, instead of being a force applied at a distance, it is the mass applied at a distance: Moment = mass(distance) from either a certain point, line, or axis.

The distance from the x-axis is given by $\frac{f(x)+g(x)}{2}$, the average of the y values from both functions, and the distance from y-axis is given by just x.

Since moment is simply mass times distance, the formulas for the moments about each axis are:

$$M_x = \rho\int_a^b \left[\frac{f(x)+g(x)}{2}\cdot(f(x)-g(x))\right]dx \quad \text{and} \quad M_y = \rho\int_a^b [x(f(x)-g(x))]dx$$

The **Center of Mass**, also known as the Centroid or Fulcrum, is the point on the planar lamina where all of the moments exactly balance. This balancing point, $(\bar{x}, \bar{y})$, is determined by the moments divided by the masses for both the x and y axes. The formulas to find this balancing point, or center of mass are:

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}$$

For a one-dimensional system (a infinitely thin rod), with uniform densities in each section of the rod, these integral equations reduce down to basic mathematics:

The equations for the Center of Mass (Centroid or Fulcrum) for a one-dimensional system are:

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$\bar{x} = \frac{M_0}{m}$, where $M_0 = m_1x_1 + m_2x_2 + m_3x_3 + ...$ and $m = m_1 + m_2 + m_3 + ...$

Classroom problems

1) Find the center of mass of a very thin, four-sectioned rod, with uniform density in each section, if the mass of each section is given by $m_1 = 8,\ m_2 = 10,\ m_3 = 3,$ and $m_4 = 7$ and the center of mass of each section is given by $x_1 = -8,\ x_2 = -4,\ x_3 = 2,$ and $x_4 = 6$.

Answer: As always, a quick sketch is very helpful for both comprehension and to use as a check to see if your final answer is reasonable. For a very thin rod, the center of mass, or fulcrum, is given by $\bar{x} = \frac{M_0}{m}$, where

$M_0 = m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4$ and $m = m_1 + m_2 + m_3 + m_4$. Simple substitution yields $M_0 = 8(-8) + 10(-4) + 3(2) + 7(6)$ and $m = 8 + 10 + 3 + 7$. Therefore, $M_0 = -56$ and $m = 28$ which yields a center of mass of $\bar{x} = -2$. A quick examination of your sketch should confirm that the point of balance of the weighted rod would be around –2 so you should feel confident in your final answer.

2) Find M_x, M_y, and the point $(\bar{x}, \bar{y})$ for a planar lamina of uniform density ρ bounded by the equations $y = \frac{-x^2}{4} + 16$ and $y = 0$.

Answer: As always, a quick sketch is very helpful for both comprehension and to use as a check to see if your final answer is reasonable. Both your sketch and your knowledge of parabolic equations should confirm that this planar lamina of uniform density is symmetric to the y-axis. Whenever a planar lamina of uniform density is symmetric to either axis, the center of mass for the OTHER variable will be zero. In other words, if a planar lamina of uniform density is symmetric to the x-axis, then $\bar{y} = 0$. If it is symmetric to the y-axis, then $\bar{x} = 0$. Since this particular planar lamina of uniform

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density is symmetric to the y-axis, $\bar{x}=0$, and, because $\bar{x}=\frac{M_y}{m}$, then M_y must also be 0. To find $\bar{y}$, you must first find M_x. Since M_x is equal to

$\rho\int_a^b\left[\frac{f(x)+g(x)}{2}\cdot(f(x)-g(x))\right]dx$, you now need to find the bound for integration.

Solving the system of equations creates $\frac{-x^2}{4}+16=0$ which, when solved, yields $x=-8$ or $x=8$. Therefore, the bounds are –8 and +8 so

$M_x=\rho\int_{-8}^{8}\left[\frac{1}{2}\left(\frac{-x^2}{4}+16+0\right)\cdot\left(\frac{-x^2}{4}+16-0\right)\right]dx\rightarrow\quad M_x=\rho\int_{-8}^{8}\left[\frac{x^4}{32}-4x^2+128\right]dx\rightarrow$

$M_x=\rho\Big|_{-8}^{8}\left(\frac{x^5}{160}-\frac{4x^3}{3}+128x\right)=\rho\left(\left(\frac{8192}{15}\right)-\left(\frac{-8192}{15}\right)\right)\rightarrow\quad M_x=\frac{16384\rho}{15}$. Now you need to find the mass, m. Recalling that $m=\rho\int_a^b\left[f(x)-g(x)\right]dx$ and using substitution yields $m=\rho\int_{-8}^{8}\left[\frac{-x^2}{4}+16-0\right]dx\rightarrow m=\rho\Big|_{-8}^{8}\left(\frac{-x^3}{12}+16x\right)\rightarrow$

$m=\rho\left(\left(\frac{256}{3}\right)-\left(\frac{-256}{3}\right)\right)=\frac{512\rho}{3}$. Therefore, since $\bar{y}=\frac{M_x}{m}$, $\bar{y}=\frac{\frac{16384\rho}{15}}{\frac{512\rho}{3}}=\frac{32}{5}$.

Therefore, the final answers are: $M_x=\frac{16384\rho}{15}$, $M_y=0$, and $(\bar{x},\bar{y})=\left(0,\frac{32}{5}\right)$.

3) Find M_x, M_y, and the point $(\bar{x},\bar{y})$ for a planar lamina of uniform density ρ bounded by the equations $y=x^2-36$ and $y=-\frac{x^3}{4}+9x$ and with a lower bounds of – 4.

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Answer: As always, a quick sketch is very helpful for both comprehension and to use as a check to see if your final answer is reasonable. For this particular problem, the sketch is critical as the two equations intersect each other THREE times. This would technically create two bounded areas that are connected by an infinitesimally small point. As this is not realistic or practical, you are provided with the lower bounds for the integration which the actual x value which separates the two areas. This means that you are only considering the area bounded by the equations when $x \geq -4$. While you were given the lower bounds for integration, you will still need to solve the system of equations to find the upper bounds. Setting the two equations equal to each other yields

$x^2 - 36 = -\frac{x^3}{4} + 9x$. Setting the equation equal to zero and clearing fractions results in

$x^3 + 4x^2 - 36x - 144 = 0$. Factoring and solving results in $x = -6,\ x = 4,$ and $x = 6$. Since – 6 is below our lower bounds, + 6 must be the upper bounds for our planar lamina.

Begin by finding the mass, *m*. Remembering that $m = \rho \int_a^b [f(x) - g(x)]dx$ and substituting yields

$$m = \rho \int_{-4}^{6} \left[\left(\frac{-x^3}{4} + 9x \right) - \left(x^2 - 36 \right) \right] dx \to \rho \int_{-4}^{6} \left[\frac{-x^3}{4} - x^2 + 9x + 36 \right] dx \to$$

$$m = \rho \Big|_{-4}^{6} \left[\frac{-x^4}{16} - \frac{x^3}{3} + \frac{9x^2}{2} + 36x \right] \to \rho \left[(-81 - 72 + 162 + 216) - \left(-16 + \frac{64}{3} + 72 - 144 \right) \right]$$

$m = \rho \left[(225) - \left(-\frac{200}{3} \right) \right] \to \frac{875\rho}{3}$. We now need to find the moment about the x-axis using the formula $M_x = \rho \int_a^b \left[\frac{f(x) + g(x)}{2} \cdot (f(x) - g(x)) \right] dx$. Substitution yields

$$M_x = \rho \int_{-4}^{6} \left[\frac{1}{2} \left(\left(-\frac{x^3}{4} + 9x \right) + \left(x^2 - 36 \right) \right) \cdot \left(\left(-\frac{x^3}{4} + 9x \right) - \left(x^2 - 36 \right) \right) \right] dx \to$$

$$\rho \int_{-2}^{1} \left[\left(\frac{-x^3}{8} + \frac{x^2}{2} + \frac{9x}{2} - 18 \right) \cdot \left(-\frac{x^3}{4} - x^2 + 9x + 36 \right) \right] dx \to$$

$$M_x = \rho\int_{-4}^{6}\left[\frac{x^6}{32}-\frac{11x^4}{4}+\frac{153x^2}{2}-648\right]dx \to \rho\Big|_{-4}^{6}\left[\frac{x^7}{224}-\frac{11x^5}{20}+\frac{51x^3}{2}-648x\right]\to$$

$$M_x = \rho\left[\left(\frac{8748}{7}-\frac{21384}{5}+5508-3888\right)-\left(\frac{-512}{7}+\frac{2816}{5}-1632+2592\right)\right]\to$$

$M_x = \rho\left[\left(-\frac{49248}{35}\right)-\left(\frac{50752}{35}\right)\right]=\frac{-20000\rho}{7}$. We also need to fine the moment about the y-axis by using $M_y = \rho\int_a^b\left[x(f(x)-g(x))\right]dx$. Substitution yields

$$M_y = \rho\int_{-4}^{6}\left[x\left(\left(-\frac{x^3}{4}+9x\right)-\left(x^2-36\right)\right)\right]dx \to \rho\int_{-4}^{6}\left[-\frac{x^4}{4}-x^3+9x^2+36x\right]dx\to$$

$$\rho\Big|_{-4}^{6}\left[-\frac{x^5}{20}-\frac{x^4}{4}+3x^3+18x^2\right]\to\rho\left[\left(-\frac{1944}{5}-324+648+648\right)-\left(\frac{256}{5}-64-192+288\right)\right]\to$$

$M_y = \rho\left[\left(\frac{2916}{5}\right)-\left(\frac{416}{5}\right)\right]=500\rho$. We now have enough information to answer the question as $\bar{x}=\frac{M_y}{m}$ and $\bar{y}=\frac{M_x}{m}$. Substitution reveals that the centroid is located at the point $\left(\frac{12}{7},-\frac{480}{49}\right)$.

4) Find the centroid of the region bounded by the equations $y=4-x^2$ and $y=x+2$.

Answer: As always, a quick sketch is very helpful for both comprehension and to use as a check to see if your final answer is reasonable. Since finding a centroid involves the evaluation of definite integrals, you must first find the bounds for the integration by solving the system of equations. In this case, you must solve $x+2=4-x^2$. Setting this quadratic equal to zero yields $x^2+x-2=0$. Factoring and solving results in the bounds of $x=1$ and $x=-2$. To find the centroid, $(\bar{x},\bar{y})$, you need to find m, M_x, and M_y.

Since $m=\rho\int_a^b\left[f(x)-g(x)\right]dx$, substitution yields

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$m=\rho\int_{-2}^{1}\left[\left(4-x^2\right)-\left(x+2\right)\right]dx$, which simplifies to $m=\rho\int_{-2}^{1}\left[-x^2-x+2\right]dx$.

Integration

yields $m=\rho\left.\left[\frac{-x^3}{3}-\frac{x^2}{2}+2x\right]\right|_{-2}^{1}\rightarrow m=\rho\left[\left(\frac{-1}{3}-\frac{1}{2}+2\right)-\left(\frac{8}{3}-2-4\right)\right]\rightarrow$

$m=\rho\left[\left(\frac{7}{6}\right)-\left(\frac{-10}{3}\right)\right]\rightarrow m=\frac{9\rho}{2}$. We now need to find the moment about the x-

axis using the formula $M_x=\rho\int_a^b\left[\frac{f(x)+g(x)}{2}\cdot(f(x)-g(x))\right]dx$. Substitution yields

$$M_x=\rho\int_{-2}^{1}\left[\frac{\left(4-x^2\right)+\left(x+2\right)}{2}\cdot\left(\left(4-x^2\right)-\left(x+2\right)\right)\right]dx\rightarrow$$

$$\rho\int_{-2}^{1}\left[\frac{-x^2+x+6}{2}\cdot\left(-x^2-x+2\right)\right]dx\rightarrow$$

$$M_x=\rho\int_{-2}^{1}\left[\frac{x^4}{2}-\frac{9x^2}{2}-2x+6\right]dx\rightarrow\rho\left.\left[\frac{x^5}{10}-\frac{3x^3}{2}-x^2+6x\right]\right|_{-2}^{1}\rightarrow$$

$M_x=\rho\left[\left(\frac{1}{10}-\frac{3}{2}-1+6\right)-\left(\frac{-16}{5}+12-4-12\right)\right]=\rho\left[\left(\frac{18}{5}\right)-\left(\frac{-36}{5}\right)\right]=\frac{54\rho}{5}$. We also

need to fine the moment about the y-axis by using $M_y=\rho\int_a^b\left[x(f(x)-g(x))\right]dx$.

Substitution yields

$$M_y=\rho\int_{-2}^{1}\left[x\left(\left(4-x^2\right)-\left(x+2\right)\right)\right]dx\rightarrow\rho\int_{-2}^{1}\left[-x^3-x^2+2x\right]dx\rightarrow\rho\left.\left[-\frac{x^4}{4}-\frac{x^3}{3}+x^2\right]\right|_{-2}^{1}\rightarrow$$

$M_y = \rho\left[\left(-\frac{1}{4}-\frac{1}{3}+1\right)-\left(-4+\frac{8}{3}+4\right)\right] = \rho\left[\left(\frac{5}{12}\right)-\left(\frac{8}{3}\right)\right] = \frac{-9\rho}{4}$. We now have enough information to answer the question as $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$. Substitution reveals that the centroid is located at the point $\left(\frac{-1}{2}, \frac{12}{5}\right)$.

5) Find M_x, M_y, and the point $(\bar{x}, \bar{y})$ for a planar lamina of uniform density ρ bounded by the equations $x = y^2 + 4y - 6$ and $x = -y^2 + 4y + 3$.

Answer: As always, a quick sketch is very helpful for both comprehension and to use as a check to see if your final answer is reasonable. From the sketch, it is obvious that the intersections of both equations complete the boundary. Setting the equations equal to each other and solving reveals that $y = \frac{3\sqrt{2}}{2}$ or $y = \frac{-3\sqrt{2}}{2}$. Substitution yields $\left(\frac{-3+12\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$ and $\left(\frac{-3-12\sqrt{2}}{2}, \frac{-3\sqrt{2}}{2}\right)$ as the points of intersection. Since this particular problem is presented as the variable *x* in terms of *y*, you can use the same formulas and simply swap out the variables. For example, the mass formula $m = \rho\int_a^b [f(x) - g(x)]dx$ can become $m = \rho\int_a^b [f(y) - g(y)]dy$. Substitution yields

$$m = \rho\int_{-3\sqrt{2}/2}^{3\sqrt{2}/2}\left[\left(-y^2+4y+3\right)-\left(y^2+4y-6\right)\right]dy \rightarrow \rho\int_{-3\sqrt{2}/2}^{3\sqrt{2}/2}\left[-2y^2+9\right]dy \rightarrow$$

$$m = \rho\Big|_{-3\sqrt{2}/2}^{3\sqrt{2}/2}\left[\frac{-2y^3}{3}+9y\right] = \rho\left[\left(\frac{-9\sqrt{2}}{2}+\frac{27\sqrt{2}}{2}\right)-\left(\frac{9\sqrt{2}}{2}-\frac{27\sqrt{2}}{2}\right)\right] \rightarrow$$

$m=\rho\left[-9\sqrt{2}+27\sqrt{2}\right]=18\sqrt{2}\rho$. We now need to find the moment about the y-axis using the formula $M_y=\rho\int_a^b\left[\frac{f(y)+g(y)}{2}\cdot(f(y)-g(y))\right]dy$. Substitution yields

$$M_y=\rho\int_{-3\sqrt{2}/2}^{3\sqrt{2}/2}\left[\frac{\left(-y^2+4y+3\right)+\left(y^2+4y-6\right)}{2}\cdot\left(\left(-y^2+4y+3\right)-\left(y^2+4y-6\right)\right)\right]dy\rightarrow$$

$$\rho\int_{-3\sqrt{2}/2}^{3\sqrt{2}/2}\left[\frac{8y-3}{2}\cdot\left(-2y^2+9\right)\right]dy\rightarrow\rho\int_{-3\sqrt{2}/2}^{3\sqrt{2}/2}\left[-16y^3+3y^2+36y-\frac{27}{2}\right]dy\rightarrow$$

$$M_y=\rho\Big|_{-3\sqrt{2}/2}^{3\sqrt{2}/2}\left[-4y^4+y^3+18y^2-\frac{27y}{2}\right]\rightarrow\rho\left[\left(-3+\frac{3\sqrt{2}}{4}+81-\frac{81\sqrt{2}}{4}\right)-\left(-3-\frac{3\sqrt{2}}{4}+81+\frac{81\sqrt{2}}{4}\right)\right]\rightarrow$$

$M_y=\rho\left[\left(78-\frac{39\sqrt{2}}{2}\right)-\left(78+\frac{39\sqrt{2}}{2}\right)\right]=-39\rho$. We also need to fine the moment about the x-axis by using $M_x=\rho\int_a^b\left[y(f(y)-g(y))\right]dy$. Substitution into this integral equation

yields $M_x=\rho\int_{-3\sqrt{2}/2}^{3\sqrt{2}/2}\left[y\left(\left(-y^2+4y+3\right)-\left(y^2+4y-6\right)\right)\right]dy\rightarrow\rho\int_{-3\sqrt{2}/2}^{3\sqrt{2}/2}\left[-2y^3+9y\right]dy\rightarrow$

$M_x=\rho\Big|_{-3\sqrt{2}/2}^{3\sqrt{2}/2}\left[-\frac{y^4}{2}+\frac{9y^2}{2}\right]\rightarrow\rho\left[\left(-\frac{81}{8}+\frac{81}{4}\right)-\left(-\frac{81}{8}+\frac{81}{4}\right)\right]=\rho\left[(0)-(0)\right]=0$. We now have enough information to answer the question as $\bar{x}=\frac{M_y}{m}$ and $\bar{y}=\frac{M_x}{m}$.

Substitution reveals that the centroid is located at the point $\left(\frac{-13\sqrt{2}}{12},0\right)$.