Teaching Notes for Calculus
Homework #3

Indeterminant Form, The Squeeze Theorem, Left & Right Sided Limits, Continuity

Teaching options: Lectures #2 is a VERY easy, if not the easiest, lecture and homework of the entire year. Lecture #3 is MUCH more difficult and VERY time consuming to both teach and to do the homework. If it is at all possible, and you have extra time during week 2, I would suggest beginning Lecture #3 so that you, and your students, don't have a meltdown during week 3.

Indeterminant form for limits: This occurs with fractional functions where the limits of both the top and the bottom of the function turn into zero at the point where x is approaching. When confronted with this form you need to try to creatively manipulate the original function by using one or more of the following tricks:

Trick #1: If you can create a function from the original function (using various mathematical manipulations) that leaves the two functions identical in their graphs EXCEPT for at the restricted value, then the limit of the modified function is the same as that of the original function.

Trick #2: Useful math manipulations include factoring and killing stuff or rationalizing the numerator. As long as the graph isn't undefined at the value x approaches, and the graph is a "normal" smooth graph without any breaks or jumps (not piecewise functions for example), then the limit of the function will just be the value obtained from substituting the value x approaches into the function.

Trick #3: You can also use the Squeeze Theorem. The Squeeze Theorem states that if you can find a function that is ALWAYS less than or equal to the function you are trying to find the limit of and you can also find another function that is ALWAYS greater or equal to the function AND the limit of both of these new function is L, then the limit of the function you couldn't find MUST be the limit of the two bounding functions, so the limit of the original function must be L!

Formally, the Squeeze Theorem states that if $g(x) \le f(x) \le h(x)$ and $\lim_{x \to a} g(x) = L$ and $\lim_{x \to a} h(x) = L$, then $\lim_{x \to a} f(x) = L$

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In order to prove this you need to recall the Definition of a Limit: $\lim_{x\to a} f(x) = L$ if, given any real number $\varepsilon > 0$, there exists another real number $\delta > 0$ so that if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$

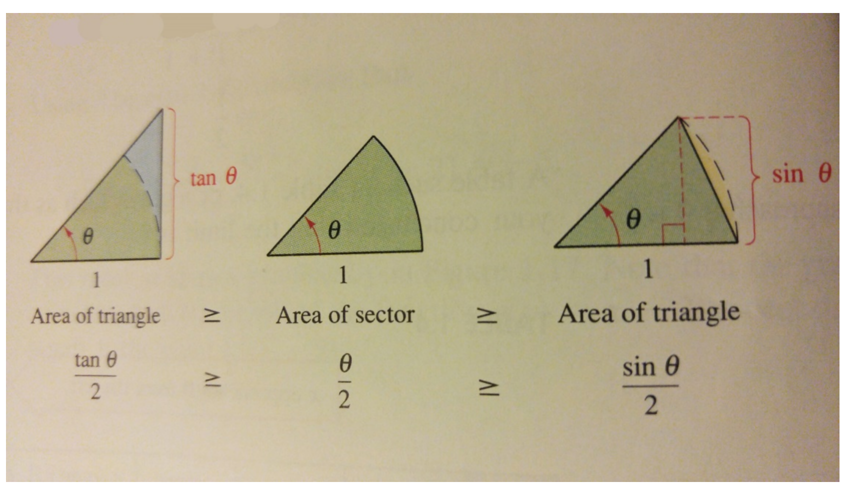
Proof of the Squeeze Theorem using the Definition of a Limit: Given: A real number $\varepsilon > 0$, $g(x) \le f(x) \le h(x)$, $\lim_{x\to a} g(x) = L$, and $\lim_{x\to a} h(x) = L$. If a $\delta > 0$ can be found such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$, then the proof is complete. If $\lim_{x\to a} g(x) = L$ then, by definition, there exists a $\delta_1 > 0$ such that when $0 < |x-a| < \delta_1$, then $|g(x) - L| < \varepsilon$. Another way to write $|g(x) - L| < \varepsilon$ is $-\varepsilon < g(x) - L < \varepsilon$. Adding L to each term results in $L - \varepsilon < g(x) < \varepsilon + L$. Using the same logic with $\lim_{x\to a} h(x) = L$ and a $\delta_2 > 0$ such that $0 < |x-a| < \delta_2$ results in $L - \varepsilon < h(x) < \varepsilon + L$. Given $g(x) \le f(x) \le h(x)$ and a $\delta_3 > 0$ such that $0 < |x-a| < \delta_3$ choose a δ for $f(x)$ that is the minimum of δ_1, δ_2, and δ_3 such that $0 < |x-a| < \delta$. Therefore, using substitution, $L - \varepsilon < g(x) \le f(x) \le h(x) < \varepsilon + L$. This implies that $L - \varepsilon < f(x) < \varepsilon + L$. Subtracting L from each term results in $-\varepsilon < f(x) - L < \varepsilon$ which can be rewritten as $|f(x) - L| < \varepsilon$, proving that $\lim_{x\to a} f(x) = L$.

It would be helpful for the students to memorize two special trig limits: $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

Prove both of these to the students: Prove the 1st one by using the Squeeze Theorem and the 2nd by using conjugates!

Proof for $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Look at the picture to the right and prove it is true. Keep in mind that everything is in radians. Multiply by 2 to get



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$\tan x \geq x \geq \sin x$ then divide everything by $\sin x$ to get $\frac{1}{\cos x} \geq \frac{x}{\sin x} \geq 1$. Now take the reciprocals and flip the signs to get $\cos x \leq \frac{\sin x}{x} \leq 1$. Since $\lim_{x \to 0} \cos x = 1$ and $\lim_{x \to 0} 1 = 1$, then, by the Squeeze Theorem, $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Proof of $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$ can be done by rationalizing the top and then using the result from the last proof…have students work this out in class!!!

The proof should look like this:

$$\lim_{x \to 0} \frac{1 - \cos x}{x} \to \lim_{x \to 0} \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x} \to \lim_{x \to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} \to$$

$$\lim_{x \to 0} \frac{\sin^2 x}{x(1 + \cos x)} \to \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x} = 1 \cdot \frac{0}{2} = 0$$

Make sure that each student gets a copy of the trig identities to use in class, on homework, and on tests. Using trig identities proves VERY helpful to determine indeterminant tips limits.

Let the students practice this concept by use trig substitutions and manipulations to try to find the limits. Remind them to be creative on both examples 1 and 2 below!

Example #1: $\lim_{x \to 0} \frac{\tan x}{x}$ since $\frac{\tan x}{x} = \frac{\sin x}{x \cos x} = \frac{\sin x}{x} \cdot \frac{1}{\cos x}$ so $\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x}$ and this equals $\lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x} = 1 \cdot 1 = 1$

Example #2: $\lim_{x \to 0} \frac{\sin 4x}{x}$ multiply and divide by 4 to get $4 \lim_{x \to 0} \frac{\sin 4x}{4x}$ then substitute using y = 4x and knowing that as x goes to 0 so does y, you to get $4 \lim_{y \to 0} \frac{\sin y}{y}$ which is 4(1) = 4

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Continuity

Definition of Continuity at a point: A function f is continuous at c if $f(c)$ is defined, $\lim_{x \to c} f(x)$ exists, and $\lim_{x \to c} f(x) = f(c)$

Definition of Continuity on an open interval: On the interval (a, b) the function must be continuous at every point in the interval

Discontinuity is the term for not being continuous and can present itself in two different ways: Removable (which can be fixed and made continuous with the addition of a single point) and Non-removable (which can't be fixed and made continuous by simply adding a single point)

One-sided Limits - $\lim_{x \to c^+} f(x)$ is the limit from the right… $\lim_{x \to c^-} f(x)$ is the limit from the left
A limit can only exist if the limit from the right equals the limit from the left

Definition of Continuity on a closed interval: In order to be continuous on the closed interval [a, b], it must be continuous on the open interval (a, b) and $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.

If two functions are continuous then the addition or subtraction or multiplication or division of the functions are also continuous

Polynomial, rational, trigonometric, absolute value, and radical functions are always continuous at every point on their domains.

If time allows, show the students this REALLY cool example of the practical use of a one-sided limit: **Charles Law and the proof of Absolute Zero**

Jacques Charles discovered that volume of gas increases with temperature and measured this expansion to come up with the formula $V = .08213T + 22.4334$ or $T = \dfrac{V - 22.4334}{.08213}$ If the volume of a gas can approach zero but never reach it, then

$$\lim_{V\to 0^+} T = \lim_{V\to 0^+} \frac{V - 22.4334}{.08213} = -273.15^o \text{ C}$$

Classroom Examples

1) Find $\lim_{x\to 3} \frac{x^3 - 27}{x - 3}$, if it exists, graph the function $f(x) = \frac{x^3 - 27}{x - 3}$ and then use this information to derive another function that agrees with $f(x)$ in all but one point.

Answer: Direct Substitution on the limit yields the indeterminate form of $\frac{0}{0}$. Therefore, you must manipulate the original function in some way to fix this issue. In this case, factoring helps as, after you factor the top, the (x – 3)'s cancel, leaving you with $f(x) = x^2 + 3x + 9$. This is the function that agrees with the original function in all but one point. Substituting 3 in for x gives you $\lim_{x\to 3}(x^2 + 3x + 9) = 27$

2) Can you find $\lim_{x\to -3} f(x)$ if $g(x) = -2x^2 - 12x - 23$, $h(x) = 3x^2 + 18x + 22$, and $g(x) \le f(x) \le h(x)$? If you can, find the limit, and explain your answer including a graph.

Answer: Yes, it can be found. $\lim_{x\to -3} f(x) = -5$ because if $\lim_{x\to -3} g(x) = -5$ and $\lim_{x\to -3} h(x) = -5$ and $g(x) \le f(x) \le h(x)$, then, according to the Squeeze Theorem, $\lim_{x\to -3} f(x)$ MUST equal –5 as illustrated in the following graph: (Create a graph with all three functions!)

3) Find $\lim_{x\to -2} \frac{x^2 + 5x + 6}{x^2 - 4}$, if it exists.

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Answer: Once again, direct substitution yields the indeterminate form of $\frac{0}{0}$. Therefore, you must manipulate the original function in some way to fix this issue. In this case, factoring helps as, after you factor the top and bottom, the (x + 2)'s cancel, leaving you with $\lim_{x \to -2} \frac{x+3}{x-2}$. This is the function that agrees with the original function in all but one point. Substituting –2 in for x gives you $\lim_{x \to -2} \frac{x+3}{x-2} = \frac{-1}{4}$

4) Find $\lim_{x \to 0} \frac{\frac{1}{x-5} + \frac{1}{5}}{2x}$, if it exists

Answer: Once again, direct substitution yields the indeterminate form of $\frac{0}{0}$. Therefore, you must manipulate the original function in some way to fix this issue. In this case, doing a jealousy game on the top give you $\frac{x}{5(x-5)}$. By writing the bottom as a fraction, $\frac{2x}{1}$, and then turning the fraction on top of fraction into multiplication to get $\frac{x}{5(x-5)} \cdot \frac{1}{2x}$ allows you to cancel the x's. This results in $\lim_{x \to 0} \frac{1}{10(x-5)}$. You can now substitute 0 for x to get $\lim_{x \to 0} \frac{1}{10(x-5)} = \frac{-1}{50}$

5) Find all values for x such that $f(x) = \frac{x-6}{x^2-36}$ is not continuous, determine which, if any, of the discontinuities are removable, and then find both the left and right sided limits for every point of discontinuity.

Answer: After factoring, the restrictions for this function are $x \neq 6, -6$. Therefore, you know that there is some type of discontinuity at each of these values. Therefore, this function is not continuous at $x = 6, -6$ After factoring, you can cancel the (x – 6)'s. Now you can graph the function to visually determine what types of discontinuities there are. The graph looks smooth at $x = 6$ which indicates a hole at 6. This means that there is a removable discontinuity at $x = 6$. Therefore, the left- and right-sided limits at 6 are $\lim_{x \to 6^-} \frac{1}{(x+6)} = \frac{1}{12}$ and $\lim_{x \to 6^+} \frac{1}{(x+6)} = \frac{1}{12}$. At $x = -6$, however, the graph is NOT smooth. It goes crazy at – 6! Approaching – 6 from the left gives you a completely different answer than approaching – 6 from the right. This means that this discontinuity is non-removable. There is a vertical asymptote at – 6. Examining the graph reveals that the left- and right-sided limits at – 6 are $\lim_{x \to -6^-} \frac{1}{(x+6)} = -\infty$ and $\lim_{x \to -6^+} \frac{1}{(x+6)} = +\infty$.