Teaching Notes for Calculus
Homework #4
The Greatest Integer Function and the Intermediate Value Theorem

$f(x) = \lfloor x \rfloor$ is the greatest integer function and means that f(x) is the largest integer less than or equal to x. *If students seem confused, do an example and sketch and quick graph that should resemble little staircase steps.*

Intermediate Value Theorem – If f(x) is continuous on the closed interval [a, b] and k is any number between f(a) and f(b), then there is at least one number c on [a, b] such that f(c) = k. *If the students seem confused, make a graph of exactly what the theorem says…it will look very similar to the graph from the first lecture.*

Classroom Examples

1) Find all values for x such that $f(x) = \lfloor x+3 \rfloor - 4$ is discontinuous. Determine which, if any, of the discontinuities are removable, and then find both the left and right sided limits at both $x = 6$ and $x = -2$.

Answers: Make a quick graph so you can physically see what is going on and where. The function is discontinuous at all integer values of x and all of the discontinuities are non-removable. $\lim_{x \to 6^+} f(x) = 5$, $\lim_{x \to 6^-} f(x) = 4$ $\lim_{x \to -2^+} f(x) = -3$, $\lim_{x \to -2^-} f(x) = -4$

2) Graph the function $f(x) = -\lfloor x \rfloor + 2x$ and then use this graph to determine all values of x at which the function is not continuous.

Answers: Make a quality graph as that is a required part of your answer and so you can physically see what is going on and where. Remember to CLEARLY show all open and closed points on the graph. The function is discontinuous at all integer values of x.

3) For the function $f(x) = -x^2 + 3x + 5$, verify that the Intermediate Value Theorem applies on the interval $-1 \le x \le 6$ and then find the value of c guaranteed by the theorem if $f(c) = -5$

Answer: $f(x) = -x^2 + 3x + 5$ is continuous on the interval [-1, 6] because it is a polynomial function. According to the Intermediate Value Theorem, if $f(-1) = 1$ and

$f(6) = -13$ then there exists at least one number c between -1 and 6, such that $f(c) =$ any real number on the interval [-13, 1] Therefore, there must be a real number, c, between –1 and 6 such that $f(c) = -5$. To find c, simply substitute –5 in for $f(x)$ to get $-5 = -c^2 + 3c + 5$ and solve for c. Therefore, $c = 5$.

4) A parcel delivery service creates delivery charges dependent on the shipping and receiving cities and based on the weight of the parcel involved. If you want to ship a parcel between Baltimore and Chicago using this delivery service, it will cost $6.20 for the first 3 pounds and then $1.15 for each additional pound or fraction thereof. Use the greatest integer function to write the cost C, of a parcel in terms of the weight, w, in pounds. Graph this function for all parcels shipped between Baltimore and Chicago weighing between 0 and 6 pounds, inclusive, find all points of discontinuity on this interval, and determine both the right and left sided limits at each point of discontinuity.

Answers: Start by writing an equation that relates the cost of sending a package to its weight. In this case, the equation would be $C(w) = \begin{cases} 0 & \text{for } w = 0 \\ 6.20 & \text{for } 0 < w \leq 3 \\ 6.20 + 1.15\lfloor w - 2 \rfloor & \text{for } 6 \geq w > 3 \end{cases}$

and graphing it.

The function is discontinuous at 0, 3, 4, 5, and 6

$\lim_{w \to 0^-} C(w) = 0$, $\lim_{w \to 0^+} C(w) = 6.20$, $\lim_{w \to 3^-} C(w) = 6.20$, $\lim_{w \to 3^+} C(w) = 7.35$,
$\lim_{w \to 4^-} C(w) = 7.35$, $\lim_{w \to 4^+} C(w) = 8.50$, $\lim_{w \to 5^-} C(w) = 8.50$, $\lim_{w \to 5^+} C(w) = 9.65$,
$\lim_{w \to 6^-} C(w) = 9.65$, $\lim_{w \to 6^+} C(w) = 10.80$

5) At 11am on Wednesday, Joshua drives his car from the base of Volcan de Agua, altitude 0, up the only road leading up to the top of the volcano. He stops several times to sightsee and for a picnic along the way up on the mountain so it takes him 6 hours to drive to the summit. He spends the night at the summit and then, at 11am on Thursday morning, he drives back down the same road to the same place where he began on Wednesday morning in only 2 hours. At some point on the way down the volcano,

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Joshua realizes that he passed the same place at exactly the same time that he did on Wednesday. Prove that Joshua must be correct by applying the Intermediate Value Theorem to his problem.

Answer: Draw a graph!!! Let s(t) be the function that represents your altitude up the mountain, assuming the base of the mountain is at an altitude of 0, and let r(t) be the function that represents your altitude as you go down the mountain. By letting 11am be t = 0 and allowing the altitude of the summit of the mountain to be k, gives s(0) = 0 and r(0) = k and s(6) = k and r(2) = 0. Now define a new function f(t) such that f(t) = s(t) – r(t). This means that f(0) = s(0) – r(0) = 0 – k < 0 and that f(2) = s(2) – r(2) = s(2) – 0 > 0. Because f(0) < 0 and f(2) > 0 there MUST be a value of t on the interval [0,2] such that f(t) = 0. If f(t) = 0 then s(t) – r(t) =0 which means that s(t) = r(t). Therefore, there MUST be at least one time, t, between 11am and 1pm inclusive where Joshua would be at the same altitude, s = r, at the same time t.

6) For the function $f(x)=\dfrac{x^2-4}{x^3-2x^2-9x+18}$, identify any discontinuities, determine which, if any are removable, and find both the left and right sided limits at all discontinuities. If the discontinuities create vertical asymptotes, find the equations of the lines.

Answers: Make a sketch of the graph to guide your conclusions and factor both the top and bottom. There are discontinuities at –2, –3, and 3. The discontinuities at –3 and 3 are non-removable, while the discontinuity at –2 is removable.
$\lim\limits_{x\to -2^-} f(x)=346$, $\lim\limits_{x\to -2^+} f(x)=346$, $\lim\limits_{x\to -3^-} f(x)=-\infty$, $\lim\limits_{x\to -3^+} f(x)=+\infty$, and $\lim\limits_{x\to 3^-} f(x)=-\infty$, $\lim\limits_{x\to 3^+} f(x)=+\infty$. The two non-removable discontinuities create vertical asymptotes with equations $x=-3$ and $x=3$.

7) For the function $f(x)=\dfrac{-x}{\cos x}$, identify any discontinuities, determine which, if any are removable, and find both the left and right sided limits at all discontinuities. If the discontinuities create vertical asymptotes, find the equations of the lines.

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Answers: There are discontinuities whenever $x = \frac{k\pi}{2}$ where k represents all odd integers. All discontinuities are non-removable and all discontinuities conform to the following pattern: For $k = 1 - 2n$ or $k = -1 + 2n$, where n is any odd natural number, then $\lim_{x \to \frac{k\pi}{2}^-} f(x) = -\infty$ and $\lim_{x \to \frac{k\pi}{2}^+} f(x) = +\infty$ and for $k = 1 - 2n$ or $k = -1 + 2n$, where n is any even natural number, then $\lim_{x \to \frac{k\pi}{2}^-} f(x) = +\infty$, $\lim_{x \to \frac{k\pi}{2}^+} f(x) = -\infty$.

Every discontinuity creates a vertical asymptote with the equation $x = \frac{k\pi}{2}$ where k represents all odd integers.

8) A 36 foot ladder is leaning against a house. If the base of the ladder is pulled away from the house at a rate of 6 feet per second, the top will move down the wall at the rate of $r = \frac{6x}{\sqrt{1296 - x^2}}$ feet per second where x is the distance the base of the ladder is from the house. Find: A. The rate when the base of the ladder is 12 feet from the house B. The rate when the base of the ladder is 27 feet from the house C. The limit of r as x approaches 36 from the left.

Answers: For Parts A & B, simply substitute the distance the base of the ladder is from the house into the rate equation. Part A: $r = \frac{3\sqrt{2}}{2} \approx 2.1$ feet per second. Part B: $r = \frac{18\sqrt{7}}{7} \approx 6.8$ feet per second. Part C: $\lim_{x \to 36^-} f(x) = +\infty$ as can be seen from the graph (substitution will not work, nor will any math manipulation!)