Teaching Notes for Calculus
Homework #5
Derivatives

Explain and demonstrate that a derivative can be thought of as slope – include a graph!

The definition of a derivative is $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$, if the limit exists.

The alternative definition of a derivative is $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$, if the limit exists. *Make sure to explain how these two definitions are basically the same!*

Two conditions that must be met for a derivative to exist at a point:

1) Both the right and left sided limits of the alternate definition of a derivative, $f'(c)$, must agree for the derivative to exist.
2) $f(x)$ must be continuous at the point of differentiation! Meaning that $f(c) = \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$ must be true!

Vertical tangent lines yield non-existent limits and undefined derivatives so you can't differentiate whenever the slope of the tangent line is undefined!

Classroom Examples

1) Use the definition of the derivative to find $f'(x)$ if $f(x) = 13$

Answer: The definition of a derivative is $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$. Using substitution, $f'(x) = \lim_{\Delta x \to 0} \frac{13 - 13}{\Delta x} = 0$. *Even though the bottom is approaching zero, it never actually becomes zero, while the top is exactly zero regardless of what's happening on the bottom. Since 0 over anything, other than 0, is 0, the derivative is 0.*

2) Find the equation of the tangent line to the graph of $f(x) = -3x^2 + 8$ at the point (2, -4) by using the definition of the derivative. Verify your answer by

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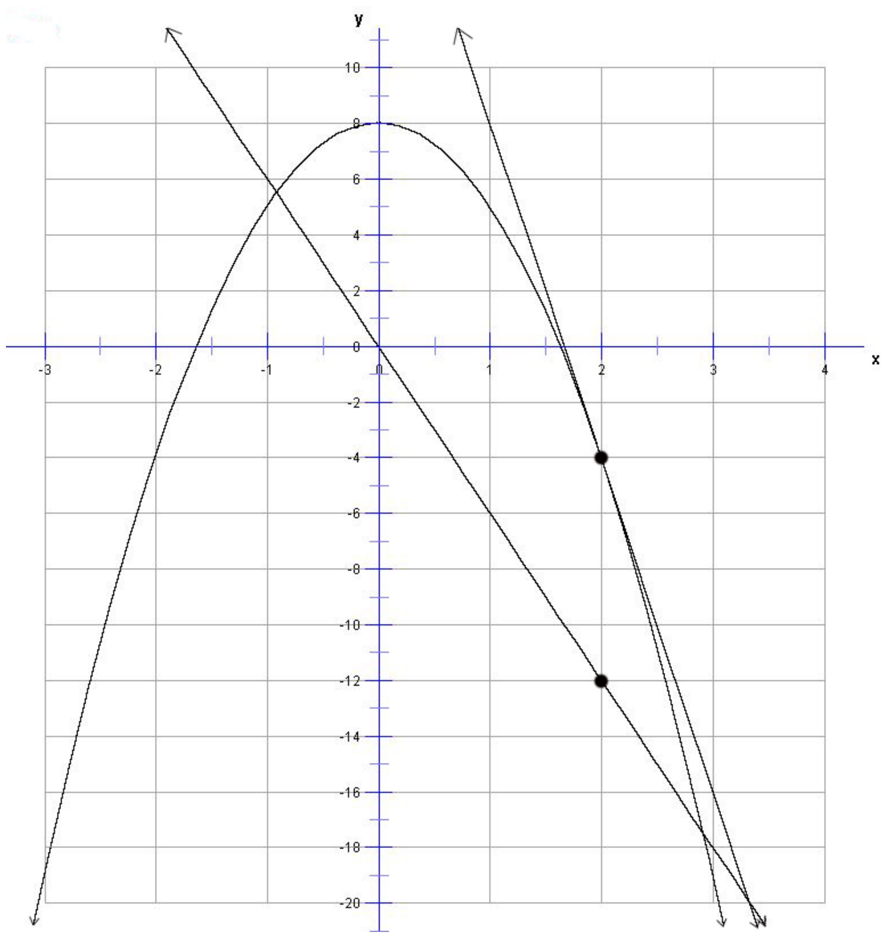
graphing $f(x)$, $f'(x)$, and the equation of the tangent line all on the same graph.

Answer: The definition of a derivative is $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$. Using substitution,

$$f'(x) = \lim_{\Delta x \to 0} \frac{(-3(x+\Delta x)^2 + 8) - (-3(x)^2 + 8)}{\Delta x} = \lim_{\Delta x \to 0} \frac{-3x^2 - 6x\Delta x - 3\Delta x^2 + 8 + 3x^2 - 8}{\Delta x} \rightarrow$$

$f'(x) = \lim\limits_{\Delta x \to 0} -6x - 3\Delta x = -6x$. That means that the equation of the tangent line is $y = -12x + b$ when x = 2. This line also goes through the point (2, -4). Substitution and solving for b gives you the equation of the tangent line, $y = -12x + 20$.



*Note that the two dots on the graph confirm that you have correctly answered the question. The point of intersection between the tangent line and the graph of the original function is where it's supposed to be, $(2, -4)$, and the y value of the point on the graph of the derivative, when x = 2, should be –12 because that is the derivative, or slope, of the original curve and, by definition, MUST be the slope of the tangent line!

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3) Use the definition of the derivative to find $f'(x)$ if $f(x) = -6x + 7$

Answer: The definition of a derivative is $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$. Using substitution,

$$f'(x) = \lim_{\Delta x \to 0} \frac{(-6(x+\Delta x)+7)-(-6(x)+7)}{\Delta x} = \lim_{\Delta x \to 0} \frac{-6x-6\Delta x+7+6x-7}{\Delta x} = -6$$

4) Use the alternate definition of the derivative to find $f'(-5)$, if it exists, if $f(x) = -x^3 + 3$

Answer: The alternate definition of a derivative is $f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$.

Finding both the left and right sided limits will prove that the limit exists and provide a formula to find it.

$$f'(c) = \lim_{x \to c^-} \frac{(-(x)^3+3)-(-(c)^3+3)}{x-c} = \lim_{x \to c^-} \frac{-x^3+3+c^3-3}{x-c} = \lim_{x \to c^-} \frac{-(x^3-c^3)}{x-c} \to$$

$$f'(c) = \lim_{x \to c^-} \frac{-(x-c)(x^2+xc+c^2)}{x-c} = \lim_{x \to c^-} (-x^2-xc-c^2) = -3c^2 \text{ and}$$

$$f'(c) = \lim_{x \to c^+} \frac{(-(x)^3+3)-(-(c)^3+3)}{x-c} = \lim_{x \to c^+} \frac{-x^3+3+c^3-3}{x-c} = \lim_{x \to c^+} \frac{-(x^3-c^3)}{x-c} \to$$

$$f'(c) = \lim_{x \to c^+} \frac{-(x-c)(x^2+xc+c^2)}{x-c} = \lim_{x \to c^+} (-x^2-xc-c^2) = -3c^2.$$

Therefore, if $f'(c) = -3c^2$ and $f(-5) = \lim\limits_{x \to -5^-} f(x) = \lim\limits_{x \to -5^+} f(x) = 128$, then the derivative exists, so $f'(-5) = -3(-5)^2 = -75$

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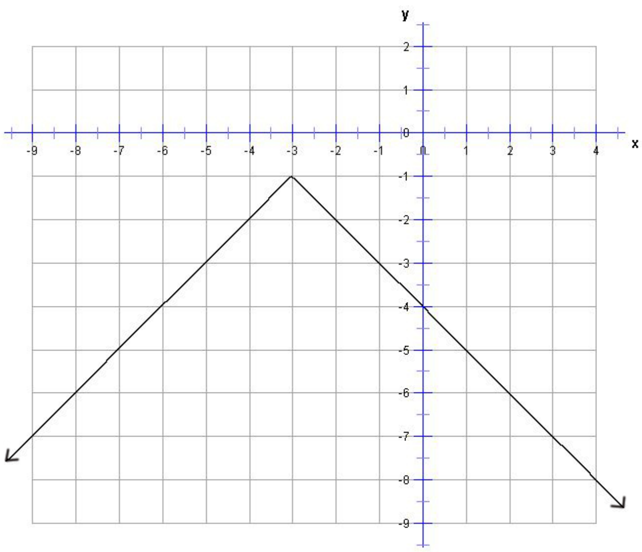
5) Find every value of x at which the function $f(x) = -2|x+3| - 1$ is differentiable. HINT: Sketch a graph of the function to help guide your conclusions.

Answer: From the graph, it is obvious that this function meets the $f(c) = \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$ for every point.



However, the final condition for differentiability is that

$f'(c) = \lim_{x \to c^-} \dfrac{f(x) - f(c)}{x - c}$ and

$f'(c) = \lim_{x \to c^+} \dfrac{f(x) - f(c)}{x - c}$ are both equal.

This is true except for all x's with the exception of $x = -3$.

$f'(-3) = \lim_{x \to -3^-} \dfrac{f(x) - f(c)}{x - c} = 1$, while

$f'(-3) = \lim_{x \to -3^+} \dfrac{f(x) - f(c)}{x - c} = -1$. Therefore, $f(x) = -2|x+3| - 1$ is differentiable for all real number, x, except for at $x = -3$.

6) Use the definition of the derivative to find $f'(x)$ if

$f(x) = -4x^2 + 3x - 7$

Answer: The definition of a derivative is $f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$. Using

substitution, $f'(x) = \lim_{\Delta x \to 0} \dfrac{(-4(x+\Delta x)^2 + 3(x+\Delta x) - 7) - (-4(x)^2 + 3(x) - 7)}{\Delta x} \rightarrow$

$$f'(x) = \lim_{\Delta x \to 0} \frac{-4x^2 - 8x\Delta x - 4\Delta x^2 + 3x + 3\Delta x - 7 + 4x^2 - 3x + 7}{\Delta x} \rightarrow \lim_{\Delta x \to 0} \frac{-8x\Delta x - 4\Delta x^2 + 3\Delta x}{\Delta x} \rightarrow$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{-8x\Delta x - 4\Delta x^2 + 3\Delta x}{\Delta x} = \lim_{\Delta x \to 0} (-8x - 4\Delta x + 3) = -8x + 3$$

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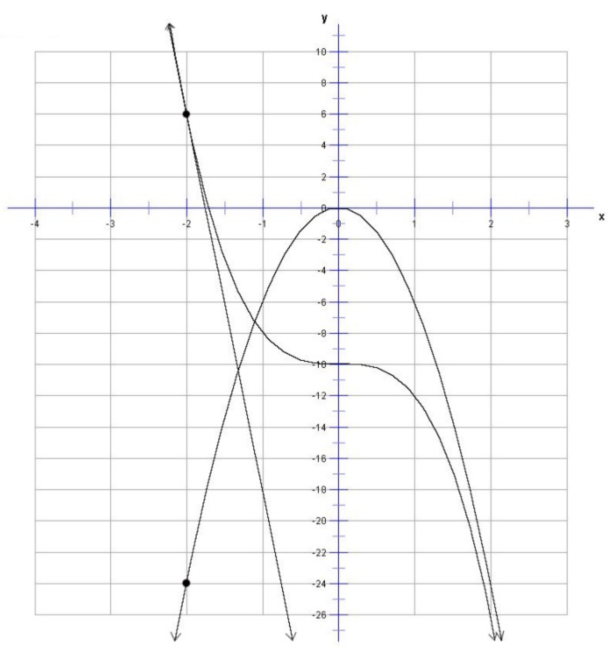
7) Find the equation of the tangent line to the graph of $f(x) = -2x^3 - 10$ at the point $(-2, 6)$ by using the definition of the derivative. Verify your answer by graphing $f(x)$, $f'(x)$, and the equation of the tangent line all on the same graph.

Answer: The definition of a derivative is $f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$. Using substitution,

$$f'(x) = \lim_{\Delta x \to 0} \frac{(-2(x+\Delta x)^3 - 10) - (-2(x)^3 - 10)}{\Delta x} = \lim_{\Delta x \to 0} \frac{-2x^3 - 6x^2\Delta x - 6x\Delta x^2 - 2\Delta x^3 - 10 + 2x^3 +}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{-6x^2\Delta x - 6x\Delta x^2 - 2\Delta x^3}{\Delta x} = \lim_{\Delta x \to 0} (-6x^2 - 6x\Delta x - 2\Delta x^2) = -6x^2,$$

therefore $f'(x) = -6x^2$ which means that $f'(-2) = -6(-2)^2 = -24$. That also means that the slope of the tangent line, at the point $(-2, 6)$ is –18. Now simply find the equation of this line as you did in Algebra I & II. Using $y = mx + b$ and substituting yields $6 = -24(-2) + b$. Solving for b, gives $b = -42$. Therefore, the equation of the tangent line is $y = -24x - 42$.



*Note that the two dots on the graph confirm that you have correctly answered the question. The point of intersection between the tangent line and the graph of the original function is where it's supposed to be, $(-2, 6)$, and the y value of the point on the graph of the derivative, when x = –2, should be –24 because that is the derivative, or slope, of the original curve and, by definition, MUST be the slope of the tangent line!

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8) Use the alternative definition of the derivative to find $f'(3)$, if it exists, if

$$f(x) = x^4 - 3x^3 - 2x^2 + 5$$

Answer: The alternate definition of a derivative is $f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$.

Finding both the left and right sided limits will prove that the limit exists and provide a formula to find it.

$$f'(c) = \lim_{x \to c^-} \frac{((x)^4 - 3(x)^3 - 2(x)^2 + 5) - ((c)^4 - 3(c)^3 - 2(c)^2 + 5)}{x - c} \rightarrow$$

$$f'(c) = \lim_{x \to c^-} \frac{x^4 - 3x^3 - 2x^2 + 5 - c^4 + 3c^3 + 2c^2 - 5}{x - c} \rightarrow$$

$$\lim_{x \to c^-} \frac{x^4 - 3x^3 - 2x^2 - c^4 + 3c^3 + 2c^2}{x - c} \rightarrow$$

$$f'(c) = \lim_{x \to c^-} \frac{\left(x^4 - c^4\right) - 3\left(x^3 - c^3\right) - 2\left(x^2 - c^2\right)}{x - c} \rightarrow$$

$$f'(c) = \lim_{x \to c^-} \frac{\left(x^2 + c^2\right)(x + c)(x - c) - 3(x - c)(x^2 + xc + c^2) - 2(x + c)(x - c)}{x - c} \rightarrow$$

$$f'(c) = \lim_{x \to c^-} \frac{(x - c)\left(\left(x^2 + c^2\right)(x + c) - 3(x^2 + xc + c^2) - 2(x + c)\right)}{x - c} \rightarrow$$

$$f'(c) = \lim_{x \to c^-} \left(\left(x^2 + c^2\right)(x + c) - 3(x^2 + xc + c^2) - 2(x + c)\right) \rightarrow$$

$$f'(c) = \left(\left(2c^2\right)(2c) - 3(3c^2) - 2(2c)\right) = 4c^3 - 9c^2 - 4c \text{ and}$$

$$f'(c) = \lim_{x \to c^+} \frac{((x)^4 - 3(x)^3 - 2(x)^2 + 5) - ((c)^4 - 3(c)^3 - 2(c)^2 + 5)}{x - c} \rightarrow$$

$$f'(c) = \lim_{x \to c^+} \frac{x^4 - 3x^3 - 2x^2 + 5 - c^4 + 3c^3 + 2c^2 - 5}{x - c} \rightarrow$$

$$\lim_{x \to c^+} \frac{x^4 - 3x^3 - 2x^2 - c^4 + 3c^3 + 2c^2}{x - c} \rightarrow$$

$$f'(c) = \lim_{x \to c^+} \frac{\left(x^4 - c^4\right) - 3\left(x^3 - c^3\right) - 2\left(x^2 - c^2\right)}{x - c} \rightarrow$$

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$$f'(c) = \lim_{x \to c^+} \frac{\left(x^2 + c^2\right)(x+c)(x-c) - 3(x-c)(x^2 + xc + c^2) - 2(x+c)(x-c)}{x-c} \to$$

$$f'(c) = \lim_{x \to c^+} \frac{(x-c)\left(\left(x^2 + c^2\right)(x+c) - 3(x^2 + xc + c^2) - 2(x+c)\right)}{x-c} \to$$

$$f'(c) = \lim_{x \to c^+} \left(\left(x^2 + c^2\right)(x+c) - 3(x^2 + xc + c^2) - 2(x+c)\right) \to$$

$f'(c) = \left(\left(2c^2\right)(2c) - 3(3c^2) - 2(2c)\right) = 4c^3 - 9c^2 - 4c$ Therefore, if

$f'(c) = 4c^3 - 9c^2 - 4c$ and $f(3) = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = -13$, then the derivative

exists, so $f'(3) = 4(3)^3 - 9(3)^2 - 4(3) = 15$