Teaching Notes for Calculus
Homework #6
Four Important Rules for Finding Derivatives & Two Trig Rules

$\frac{d}{dx}(x)^n = nx^{n-1}$ **: This is the Power Rule** - *Demonstrate this by showing that the limit definition of the derivative always gives the answer from the power rule! Start with the definition of a derivative, $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$. If $f(x) = (x)^n$ and $f(x+\Delta x) = (x+\Delta x)^n$ then $f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - (x)^n}{\Delta x}$. Now, think about the expansion of a binomial to the nth degree: $(x+\Delta x)^1 \to x^1 + \Delta x$, $(x+\Delta x)^2 \to x^2 + 2x\Delta x + \Delta x^2$, $(x+\Delta x)^3 \to x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3$, and $(x+\Delta x)^4 \to x^4 + 4x^3\Delta x + 6x^2\Delta x^2 + 4x\Delta x^3 + \Delta x^4$, etc. What do you notice? If you separate the very first term from each of these expansions, separating the $(x)^n$ term, all of the remaining terms contain at least one Δx. Now notice that the top of the definition of a derivative ends in $-(x)^n$. This means that the $(x)^n$ that you separated, combined with the $-(x)^n$ that's always at the end, cancel each other out. That results in at least one Δx in EVERY term on the top. Factoring out that one Δx, allows you to cancel the Δx on the bottom. Now look at what's left on the top. Your final answer for the derivative depends on what n was. If n was 2, the only term remaining without a Δx is $2x$. If n was 3, the only term remaining without a Δx is $3x^2$. If n was 4, the only term remaining without a Δx is $4x^3$. Notice a pattern? If $f(x) = (x)^2$, then $f'(x) = 2x^1$. If $f(x) = (x)^3$, then $f'(x) = 3x^2$. If $f(x) = (x)^4$, then $f'(x) = 3x^4$. In each function, the power on x became the number that x is multiplied by, and the exponent got reduced by 1. Therefore, $f'((x)^n) = nx^{n-1}$.

$\frac{d}{dx}(f(x) \cdot g(x)) = g(x) \cdot f'(x) + f(x) \cdot g'(x)$ **- This is the Product Rule** - *Prove this with the limit definition of a derivative* Proof: Start by stating that $\frac{d}{dx}(f(x) \cdot g(x))$ can be written as $(f \cdot g)'(x)$. Apply the definition of a derivative, $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$, and substitute $f \cdot g$ in for f which yields $(f \cdot g)'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) \cdot g(x+\Delta x) - f(x) \cdot g(x)}{\Delta x}$.

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You can then subtract and add $f(x+\Delta x)g(x)$ in between the two terms in the numerator because you are essentially adding 0. This gives you $(f\cdot g)'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)\cdot g(x+\Delta x)-f(x+\Delta x)\cdot g(x)+f(x+\Delta x)\cdot g(x)-f(x)\cdot g(x)}{\Delta x}$. Factoring the top gives you $(f\cdot g)'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)(g(x+\Delta x)-g(x))+g(x)(f(x+\Delta x)-f(x))}{\Delta x}$. Split this sum into 2 limits $(f\cdot g)'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)(g(x+\Delta x)-g(x))}{\Delta x}+\lim_{\Delta x\to 0}\frac{g(x)(f(x+\Delta x)-f(x))}{\Delta x}$. Now split the products to get $(f\cdot g)'(x)=\lim_{\Delta x\to 0}f(x+\Delta x)\cdot\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}+\lim_{\Delta x\to 0}g(x)\cdot\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$.
Using the definition of a derivative and taking the limits gives you
$(f\cdot g)'(x)=f(x)\cdot g'(x)+g(x)\cdot f'(x)$ which can also be written as
$(f\cdot g)'(x)=g(x)\cdot f'(x)+f(x)\cdot g'(x)$

$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\cdot f'(x)-f(x)\cdot g'(x)}{(g(x))^2}$ - **This is the Quotient Rule** - *Prove this rule by forcing g(x) up and using the product rule! So,

$\left(\frac{f(x)}{g(x)}\right)'=(f\cdot g^{-1})'(x)=g^{-1}(x)\cdot f'(x)-f(x)\cdot g^{-2}(x)\cdot g'(x)$ which simplifies to

$\left(\frac{f(x)}{g(x)}\right)'=\frac{f'(x)}{g(x)}-\frac{f(x)\cdot g'(x)}{g^2(x)}=\frac{g(x)\cdot f'(x)-f(x)\cdot g'(x)}{g^2(x)}$ *The students need to wait for Worksheet #7 to completely understand where all the $g'(x)$'s came from so don't get hung up on that*

$\frac{d}{dx}\left(\frac{1}{g(x)}\right)=\frac{-g'(x)}{(g(x))^2}$ - **This is the Reciprocal Rule** - *The reciprocal rule is derived from the quotient rule – it's not necessary to know or memorize, but makes things go quicker!

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First two trig rules for derivatives:

Give the students the $\frac{d}{dx}(\sin x) = \cos x$ derivative trig rule and the $\frac{d}{dx}(\cos x) = -\sin x$ rule. Demonstrate how each rule makes sense by graphing sin(x) and cos(x) and showing that the slope of sin(x) always equals cos(x). By the same method, show that the slope of cos(x) always equals –sin(x)…especially at 90 degrees!

Classroom Examples

1) Determine the points, if any, at which the function $y = x^5 - 160x^2$ has a horizontal tangent line.

Answer: $y' = 5x^4 - 320x = 0$ $5x(x^3 - 64) = 0$ $5x(x-4)(x^2 + 4x + 16) = 0$ (0,0) & (4,-1536)

2) Find the equation of the line tangent to $y = x^3 + 4x^2 + 3x - 5$ at the point $(-2, -3)$

Answer: $y' = 3x^2 + 8x + 3$ $y'(-2) = 3(-2)^2 + 8(-2) + 3 = -1$ $y = -1x + b$
$(-3) = -1(-2) + b$ $b = -5$ Therefore the tangent line is $y = -1x - 5$

3) Find $f'(x)$ if $f(x) = (x^3 - 4x^2 + 5)(-x^3 + 3x - 2)$ and then calculate $f'(-3)$

Answer: Call $h(x) = x^3 - 4x^2 + 5$ and $g(x) = -x^3 + 3x - 2$ Use the product rule to find

$$(h(x) \cdot g(x))' = g(x) \cdot h'(x) + h(x) \cdot g'(x) = (-x^3 + 3x - 2)(3x^2 - 8x) + (x^3 - 4x^2 + 5)(-3x^2 + 3$$

$$(h(x) \cdot g(x))' = (-x^3 + 3x - 2)(3x^2 - 8x) + (x^3 - 4x^2 + 5)(-3x^2 + 3)$$
$$(h(x) \cdot g(x))' = -6x^5 + 20x^4 + 12x^3 - 57x^2 + 16x + 15 = f'(x) \qquad f'(-3) = 2208$$

4) Find $f'(x)$ if $f(x) = \dfrac{-4x - 7}{3x - 1}$ and find $g'(x)$ if $g(x) = \dfrac{1}{3x^2 - 5x}$

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Answer #1: Call $h(x) = -4x - 7$ and $g(x) = 3x - 1$ Use the quotient rule to find

$$\left(\frac{h(x)}{g(x)}\right)' = \frac{g(x)\cdot h'(x) - h(x)\cdot g'(x)}{\left(g(x)\right)^2} = \frac{(3x-1)(-4)-(-4x-7)(3)}{(3x-1)^2} = \frac{25}{9x^2-6x+1}$$

Answer #2: Call $h(x) = 3x^2 - 5x$ Use the reciprocal rule to find

$$\left(\frac{1}{h(x)}\right)' = \frac{-h'(x)}{\left(h(x)\right)^2} = \frac{6x-5}{(3x^2-5x)^2} = \frac{6x-5}{9x^4-30x^3+25x^2}$$

5) Find $f'(x)$ if $f(x) = (-5x+3)^2$

Answer: $f(x) = 25x^2 - 30x + 9$ so $f'(x) = 50x - 30$

6) Find $g'(x)$ if $g(x) = \dfrac{4x}{-3+5\sin x}$

Answer: Call $h(x) = 4x$ and $f(x) = -3 + 5\sin x$ Use the quotient rule to find

$$\left(\frac{h(x)}{f(x)}\right)' = \frac{f(x)\cdot h'(x) - h(x)\cdot f'(x)}{\left(f(x)\right)^2} = \frac{(-3x+5\sin x)(4)-(4x)(5\cos x)}{(-3+5\sin x)^2} = \frac{-12x+20\sin x-20x\cos x}{25\sin^2 x - 30\sin x + 9}$$

7) Find the derivative of $f(x) = \dfrac{\sqrt[4]{x^3}}{x^2}$

Answer: $f(x) = \dfrac{x^{\frac{3}{4}}}{x^2} = \dfrac{1}{x^{\frac{5}{4}}}$ Call $h(x) = x^{\frac{5}{4}}$ Use the reciprocal rule to find

$$\left(\frac{1}{h(x)}\right)' = \frac{-h'(x)}{\left(h(x)\right)^2} = \frac{-\frac{5}{4}x^{\frac{1}{4}}}{(x^{\frac{5}{4}})^2} = \frac{-\frac{5x^{\frac{1}{4}}}{4}}{x^{\frac{5}{2}}} = -\frac{5x^{\frac{1}{4}}}{4x^{\frac{5}{2}}} = \frac{-5}{4x^{\frac{9}{4}}} = \frac{-5}{4\sqrt[4]{x^9}} = \frac{-5\sqrt[4]{x^3}}{4x^3}$$

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8) Find the derivative of $h'(x)$ if $h(x)=\dfrac{2-\sin x}{3+2\cos x}$

Answer: Call $g(x)=2-\sin x$ and $f(x)=3+2\cos x$ Use the quotient rule to find

$$\left(\frac{g(x)}{f(x)}\right)'=\frac{f(x)\cdot g'(x)-g(x)\cdot f'(x)}{\left(f(x)\right)^2}=\frac{(3+2\cos x)(-\cos x)-(2-\sin x)(-2\sin x)}{(3+2\cos x)^2}=$$

$$\left(\frac{g(x)}{f(x)}\right)'=\frac{-3\cos x-2\cos^2 x+4\sin x-2\sin^2 x}{4\cos^2 x+12\cos x+9}=\frac{4\sin x-3\cos x-2}{4\cos^2 x+12\cos x+9}$$