Teaching Notes for Calculus
Homework #8
Higher Order Derivatives and Implicit Differentiation

Higher order derivatives can go on endlessly. It simply means that if you start with a function, $f(x)$, its 1st derivative is $f'(x)$. Its 2nd derivative is $f''(x)$, its 3rd derivative is $f'''(x)$, etc.

This is applicable to the real world in that the derivative of distance is velocity, derivative of velocity is acceleration. Therefore, the 2nd derivative of distance is acceleration.

Example: Suppose you go sky diving. Neglecting wind resistance, and assuming you haven't opened your parachute yet, you can determine lots of information about what's happening to you by just knowing the formula for the distance you've fallen. The formula is $d(t) = 16t^2$ where t is in seconds and distance traveled is measured in feet. Using differentiation, the derivative of distance is velocity so if $d(t) = 16t^2$, then $v(t) = 32t$. If the derivative of velocity is acceleration, then $a(t) = 32$. This means that 5 seconds after jumping out of the airplane, you would be accelerating at $a(5) = 32$ feet per second per second. Your velocity would be $v(5) = 32(5) = 160$ feet per second and the distance you traveled would be $d(5) = 16(5)^2 = 400$ feet. We will explore this concept, in depth, during our next lesson.

So far, every derivative problem encountered in this course has had y alone. Doing a derivative on this type of problem is called explicit differentiation.

Implicit differentiation is what you need to do when y isn't alone and it would be extremely difficult, or impossible, to get y alone. *Note: You can clear fractions before you differentiate and you will get the correct answer…it just won't look like what you get from differentiating first, but the answers are equivalent.*

Examples: Find the derivative of $3x^2 - y^2 = 5x - 4y$ using implicit differentiation.

Answer: $6x \cdot \frac{dx}{dx} - 2y \cdot \frac{dy}{dx} = 5 \cdot \frac{dx}{dx} - 4 \cdot \frac{dy}{dx} \rightarrow 4 \cdot \frac{dy}{dx} - 2y \cdot \frac{dy}{dx} = 5 - 6x \rightarrow$

Teaching Notes for Calculus
Homework #8

$\frac{dy}{dx}(4-2y)=5-6x \rightarrow \frac{dy}{dx}=\frac{5-6x}{4-2y}$ *This is the most rigorous but easiest method to understand…once you REALLY understand this method you can try the following method:

Find the derivative of $x^2+4y^2=6x^3y$ using implicit differentiation.

Answer: $2x+8yy'=18x^2y+6x^3y' \rightarrow 2x-18x^2y=6x^3y'-8yy' \rightarrow$

$$2x-18x^2y=2y'(3x^3-4y) \rightarrow y'=\frac{2x(1-9xy)}{2(3x^3-4y)} \rightarrow y'=\frac{x(1-9xy)}{3x^3-4y}$$

Classroom Exercises:

1) Use implicit differentiation to find $\frac{dy}{dx}$ if $x^4-5y^3=-7x^2$

Answer: $4x^3\cdot\frac{dx}{dx}-15y^2\cdot\frac{dy}{dx}=-14x\cdot\frac{dx}{dx}\rightarrow -15y^2\cdot\frac{dy}{dx}=-14x-4x^3\rightarrow\frac{dy}{dx}=\frac{14x+4x^3}{15y^2}$

2) If $f(x)=2x^3-3x+9$, find $f'(x)$, $f''(x)$, and $f'''(x)$

Answer: $f'(x)=6x^2-3$ $\quad f''(x)=12x$ $\quad f'''(x)=12$

3) Use implicit differentiation to find $\frac{dy}{dx}$ if $7x^3y^4-3y^2x^3=8x^5$

Answer: $21x^2y^4\cdot\frac{dx}{dx}+28x^3y^3\cdot\frac{dy}{dx}-6yx^3\cdot\frac{dy}{dx}-9y^2x^2\cdot\frac{dx}{dx}=40x^4\cdot\frac{dx}{dx}\rightarrow$

$$28x^3y^3\cdot\frac{dy}{dx}-6yx^3\cdot\frac{dy}{dx}=40x^4+9y^2x^2-21x^2y^4\rightarrow\frac{dy}{dx}\cdot(28x^3y^3-6yx^3)=40x^4+9y^2x^2-21x^2y^4\rightarrow$$

$$\frac{dy}{dx}=\frac{x^2(40x^2+9y^2-21y^4)}{2x^3y(14y^2-3)}\rightarrow\frac{40x^2+9y^2-21y^4}{2xy(14y^2-3)}$$

Teaching Notes for Calculus
Homework #8

4) Use implicit differentiation to find $\frac{dy}{dx}$ if $-3\tan x \csc y = 5\sin y$

Answer: $-3\sec^2 x \cdot \csc y \cdot \frac{dx}{dx} + 3\csc y \cdot \cot y \cdot \tan x \cdot \frac{dy}{dx} = 5\cos y \cdot \frac{dy}{dx} \rightarrow$

$-3\sec^2 x \cdot \csc y = 5\cos y \cdot \frac{dy}{dx} - 3\csc y \cdot \cot y \cdot \tan x \cdot \frac{dy}{dx} \rightarrow$

$-3\sec^2 x \cdot \csc y = \frac{dy}{dx}(5\cos y - 3\csc y \cdot \cot y \cdot \tan x) \rightarrow \frac{dy}{dx} = \frac{-3\sec^2 x \cdot \csc y}{5\cos y - 3\csc y \cdot \cot y \cdot \tan x}$

5) Use implicit differentiation to find $\frac{dy}{dx}$ if $x + y\sin y = \sec(3y^2 - 5y)$

Answer: $\frac{dx}{dx} + \frac{dy}{dx}\sin y + \frac{dy}{dx} \cdot y \cdot \cos y = \left(6y \cdot \frac{dy}{dx} - 5 \cdot \frac{dy}{dx}\right) \cdot \sec(3y^2 - 5y) \cdot \tan(3y^2 - 5y) \rightarrow$

$1 + \frac{dy}{dx}(\sin y + y \cdot \cos y) = \frac{dy}{dx}(6y - 5) \cdot \sec(3y^2 - 5y) \cdot \tan(3y^2 - 5y) \rightarrow$

$\frac{dy}{dx}(\sin y + y \cdot \cos y) - \frac{dy}{dx}(6y - 5) \cdot \sec(3y^2 - 5y) \cdot \tan(3y^2 - 5y) = -1 \rightarrow$

$\frac{dy}{dx}\left(\sin y + y \cdot \cos y - (6y - 5) \cdot \sec(3y^2 - 5y) \cdot \tan(3y^2 - 5y)\right) = -1 \rightarrow$

$$\frac{dy}{dx} = \frac{-1}{\sin y + y \cdot \cos y - (6y - 5) \cdot \sec(3y^2 - 5y) \cdot \tan(3y^2 - 5y)}$$

6) Use implicit differentiation to find $\frac{dy}{dx}$ if $-7 - 3y^3 = \left(\frac{y^2 - 5}{x + 2}\right)$ and then evaluate the derivative at the point $(-1, -1)$.

Answer: $-9y^2 \cdot \frac{dy}{dx} = -\frac{dx}{dx} \cdot \left(\frac{y^2 - 5}{(x+2)^2}\right) + \frac{dy}{dx}\left(\frac{2y}{x+2}\right) \rightarrow \frac{y^2 - 5}{(x+2)^2} = \frac{dy}{dx}\left(\frac{2y}{x+2}\right) + 9y^2 \cdot \frac{dy}{dx} \rightarrow$

$\frac{y^2 - 5}{(x+2)^2} = \frac{dy}{dx}\left(\frac{2y}{x+2} + 9y^2\right) \rightarrow \frac{y^2 - 5}{(x+2)^2} = \frac{dy}{dx} \cdot \left(\frac{2y + 9y^2x + 18y^2}{x+2}\right) \rightarrow \frac{dy}{dx} = \frac{(y^2 - 5)(x+2)}{(x+2)^2(2y + 9y^2x + 18y^2)}$

$$\frac{dy}{dx}(-1, -1) = \frac{-4}{7}$$