Teaching Notes for Calculus
Homework #9
Average Velocity, Instantaneous Velocity, and Related Rates

Average Velocity is the change in distance divided by the change in time over a given interval.

Average Rate works the same way, just with different, or no, units.

*Note – While many textbooks use *d* to represent distance, there are more, especially in physics, that use the letter *s* to represent distance.*

Instantaneous Velocity is the velocity at a given instant or

$$v(t) = \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t} = s'(t).$$

Instantaneous Rate works the same way, just with different, or no, units

Remember, from the last lecture, that velocity is the derivative of distance and acceleration is the derivative of velocity. Also, you need to remember from either algebra II, pre-calculus, or physics, that v_0 is the initial velocity and s_0 is the initial height in the gravitational distance function on planet Earth

Related Rates occur when you have multiple variables that all depend on other variables. This is especially applicable to solving difficult, practical word problems where the rate of change of two or more objects depends on each other. For example, if you have 3 variables in an equation, *x*, *y*, and *z*, and they all depend on *t*, or they are all differentiable functions of *t,* you could be asked to find $\frac{dx}{dt}$ or/and $\frac{dy}{dt}$ or/and $\frac{dz}{dt}$.

Classroom Examples:

1) Find the average rate of change of the function $f(t) = -t^2 + 8t + 6$ over the interval $[3,5]$. Compare this average rate of change to the instantaneous rates of change at the endpoints of the interval.

Answers: Average Rate: $\frac{f(5) - f(3)}{5-3} = 0$ Instantaneous Rates: $f'(t) = -2t + 8$ therefore, $f'(3) = -2(3) + 8 = 2$ while $f'(5) = -2(5) + 8 = -2$

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2) Assuming that x and y are both differentiable functions of t in the equation $3y^2 = -4x^3 + 5x^2 - 3x + 2$, find $\frac{dy}{dt}$ when x = 2, y = – 5, and $\frac{dx}{dt} = -3$. Find $\frac{dx}{dt}$ when x = –4, y = 3, and $\frac{dy}{dt} = 6$.

Answers: Differentiate with respect to t: $6y \cdot \frac{dy}{dt} = -12x^2 \cdot \frac{dx}{dt} + 10x \cdot \frac{dx}{dt} - 3 \cdot \frac{dx}{dt}$ then substitute 2 for x, –5 for y, and –3 for $\frac{dx}{dt}$, simplify, and solve for $\frac{dy}{dt}$. This gives you $\frac{dy}{dt} = \frac{-31}{10}$. For $\frac{dx}{dt}$, substitute –4 for x, 3 for y, and 6 for $\frac{dy}{dt}$, simplify, and solve for $\frac{dx}{dt}$. This gives you $\frac{dx}{dt} = \frac{-18}{235}$.

3) A ball is thrown straight down from the top of a 340 foot tall building with an initial velocity of –35 feet per second. What is its velocity after 2 seconds, what is its velocity after falling 249 feet, and what is the acceleration if the position function for any free-falling object on Earth is $s(t) = -16t^2 + v_0 t + s_0$.

Answers: If $s(t) = -16t^2 + v_0 t + s_0$, then $v(t) = s'(t) = -32t + v_0$, and $a(t) = v'(t) = -32$, then when $v_0 = -35$ and $t = 2$, $v(2) = -32(2) - 35 = -99$ so the velocity after 2 seconds is –99 feet per second.

If the ball falls 249 feet, then $s(t) = -16t^2 - 35t + 340 = 91$. Solving by factoring or quadratic formula gives $t = -\frac{83}{16}$ or 3. Since time cannot be negative, the only realistic answer is that $t = 3$. If $t = 3$, then $v(3) = -32(3) - 35$ which means that the velocity is –131 feet per second after the ball has fallen 249 feet.

The acceleration due to gravity is $a(t) = -32$ which means that the ball's acceleration is constant, regardless of the time. Therefore, the ball's acceleration is –32 feet per second squared.

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4) The radius, r, of a right cone with height 12 is increasing at a rate of 3 inches per minute. Find the rate of change of the volume when r = 4 inches and then again when r = 16 inches. Explain why the rate of change of the volume of the cone is not constant even though $\frac{dr}{dt}$ is constant.

Answers: Since the volume of a right cone is $V = \frac{1}{3}\pi r^2 h$, and we are looking for the rate of change of the volume with respect to time, we will use implicate differentiations, remembering that the height of the cone is constant. Therefore, $\frac{dV}{dt} = \frac{2}{3}\pi \cdot r \cdot h \cdot \frac{dr}{dt}$. If $r = 4$, $h = 12$, and $\frac{dr}{dt} = 3$, then $\frac{dV}{dt} = 48\pi$ inches cubed per minute. If $r = 16$, $h = 12$, and $\frac{dr}{dt} = 3$, then $\frac{dV}{dt} = 384\pi$ inches cubed per minute. The change in the volume is not only dependent on $\frac{dr}{dt}$, but also on r itself. That's why $\frac{dV}{dt}$ is constant.

5) Verify that the average velocity over the time interval $\left[t_0 - \Delta t, t_0 + \Delta t\right]$ is the same as the instantaneous velocity at $t = t_0$ for the position function $s(t) = -3mt^2 + k$

Answer: The average velocity is
$\frac{\Delta s}{\Delta t} = \frac{(-3m \cdot (t_0 + \Delta t)^2 + k) - (-3m \cdot (t_0 - \Delta t)^2 + k)}{(t_0 + \Delta t) - (t_0 - \Delta t)} = \frac{-12mt_0 \Delta t}{2\Delta t} = -6mt_0$. The instantaneous velocity is $s'(t) = -6mt_0$

6) A ladder 65 feet long is leaning against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 6 feet per second. How fast is the top moving down the wall when the base of the ladder is 33 feet, 52 feet, and 56 feet from the wall? Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 33 feet from the wall? Find the rate at which the angle between the top of the ladder and the wall of the house is changing when the base of the ladder is 33 feet from the wall.

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Answers: Draw a picture of a ladder with one end on the ground and the other end leaning up against a wall so that the length of the ladder, the distance from the base of the building to the bottom of the ladder, and the distance from the base of the building to the top of the ladder form a right triangle. Call the base, x, the wall height, H, and the length of the ladder s. Use the Pythagorean Theorem to get $x^2 + H^2 = s^2$ and then implicitly differentiate, with respect to time, to get $2x \cdot \frac{dx}{dt} + 2H \cdot \frac{dH}{dt} = 2s \cdot \frac{ds}{dt}$. Use the Pythagorean Theorem to find unknown lengths, remembering that the length of the ladder never changes, and solving for $\frac{dH}{dt}$ for each value of the length of the base gives $\frac{dH}{dt}(33) = \frac{-99}{28}\frac{\text{ft}}{\text{sec}}$, $\frac{dH}{dt}(52) = \frac{-8}{1}\frac{\text{ft}}{\text{sec}}$, and $\frac{dH}{dt}(56) = \frac{-112}{11}\frac{\text{ft}}{\text{sec}}$. To find the change in area as the ladder slides down the wall, you need to use answers from the first question, along with the knowledge that $A = \frac{1}{2}x \cdot h$ which implicitly differentiates into $\frac{dA}{dt} = \frac{1}{2} \cdot x \cdot \frac{dh}{dt} + \frac{1}{2} \cdot h \cdot \frac{dx}{dt}$. Using substitution, you get the answer $\frac{dA}{dt}(33) = \frac{6141}{56}\frac{\text{ft}^2}{\text{sec}}$. Finally, to find the rate of change of the angle between the top of the ladder and the wall, you need to use trigonometry to find the relationship for α, $\tan\alpha = \frac{x}{H}$. Differentiating with respect to time yields $\frac{d\alpha}{dt} \cdot \sec^2\alpha = \frac{-x}{H^2} \cdot \frac{dH}{dt} + \frac{1}{H} \cdot \frac{dx}{dt}$. Knowing that this is a 33, 56, 65 right triangle, you could write $\tan\alpha = \frac{33}{56}$, which could be write $\cos\alpha = \frac{56}{65}$ or $\frac{1}{\cos\alpha} = \frac{65}{56}$, or $\frac{1}{\cos^2\alpha} = \left(\frac{65}{56}\right)^2$, and finally as $sec^2\alpha = \left(\frac{65}{56}\right)^2$. Substituting all of the known values into the differential equation gives you $\frac{d\alpha}{dt} \cdot \left(\frac{65}{56}\right)^2 = \frac{-33}{56^2} \cdot \frac{-99}{28} + \frac{1}{56} \cdot 6$. Solving for $\frac{d\alpha}{dt}(33)$ gives you $\frac{d\alpha}{dt}(33) = \frac{3}{28}\frac{\text{rad}}{\text{sec}}$

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7) An air traffic controller spots two planes at the same altitude converging on a point as they fly at right angles to each other. One plane is 175 miles from the point and is moving at 700 miles per hour. The other plane is 420 miles from the point and has a speed of 1680 miles per hour. At what rate is the distance, s, between the planes decreasing? How much time does the traffic controller have to get one of the planes on a different flight path?

Answers: Draw a picture of the two planes. Use the origin of the Cartesian graph as the point of collision and draw the first airplane down on the negative x axis, headed towards the origin and the second plan up on the positive y axis, headed towards the origin. Construct a right triangle between the two planes and the origin. Call the distance the first plane is away from the origin, x, the distance the second plane is away from the origin, y, and the distance between the two planes, s. Use the Pythagorean Theorem to get $x^2 + y^2 = s^2$. Implicitly differentiate with respect to time yields $2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 2s \cdot \frac{ds}{dt}$. The problem gives you all of these values expect for $\frac{ds}{dt}$ so simply substitute and solve to get $\frac{ds}{dt} = \frac{-43820 \text{ miles}}{13 \text{ hour}}$ *Remember that both planes are headed towards the origin so there speeds are both negative!* To find the time limit for the planes to be redirected to avoid the crash, you need to find the time it will take the distance between the planes, s, to go from 455 miles (use the Pythagorean Theorem to find this number) down to 0. For this, you can use D =RT where D = -455 miles and $R = \frac{ds}{dt} = \frac{-43820 \text{ miles}}{13 \text{ hour}}$ to solve for T

$$\text{Time}_{hrs} = \frac{169}{1252} \text{ hours which is } \text{Time}_{\min} = \frac{2535}{313} \text{ minutes} \approx 8.1 \text{ minutes}$$

8) A woman 5 feet tall walks at a rate of 3 feet per second away from a light that is 12 feet above the ground. When she is 18 feet from the base of the light, at what rate is the tip of her shadow changing and at what rate is the length of her shadow changing?

Answers: Problems like this require a detailed picture. Drawing the ground, the pole with the light on it, the person who is walking away from the pole, and by drawing a line from the light to the top of the person's head and continue until the line hits the ground, completes the picture. Call the distance between the pole and the person, x. Call the distance from the pole to the tip of the head of the person's shadow, S. Recognize that the two triangles formed are similar due to AAA similarity. This means that the matching sides are proportional. Therefore, $\frac{12}{S} = \frac{5}{S-x}$. Clearing fractions yields $12S - 12x = 5S$, which simplifies to $12x = 7S$. Implicitly differentiate with respect to time yields $12 \cdot \frac{dx}{dt} = 7 \cdot \frac{dS}{dt}$. Substituting 3 for $\frac{dx}{dt}$ and solving for $\frac{ds}{dt}$ gives you $\frac{ds}{dt} = \frac{36}{7} \frac{\text{ft}}{\text{sec}}$, which is the rate of change of the tip of her shadow. To find the rate of change of the length of the shadow requires you to realize that the length of the shadow, L, is S – x. Therefore the rate of change of L must equal the difference between the rates of change of S and x. This results in a final answer of $\frac{dL}{dt} = \frac{ds}{dt} - \frac{dx}{dt} = \frac{15}{7} \frac{\text{ft}}{\text{sec}}$